With these assumptions / scalings, we have

$$\boxed{\frac{\partial}{\partial t}\left(\nabla^2 \psi + \frac{f_0^2}{N^2}\frac{\partial^2 \psi}{\partial z^2}\right) + \beta_0 \frac{\partial \psi}{\partial x} = 0}$$

seek solutions $\psi(x,y,z,t) = \underline{a(z)}\ \underline{\cos(kx + ly - \omega t)}$

which gives

$$\boxed{\frac{d^2 a}{dz^2} - \left(\frac{N}{f_0}\right)^2\left(k^2 + l^2 + \frac{\beta_0 k}{\omega}\right) \cdot a = 0}$$

This is an ordinary differential equation for the vertical structure $a(z)$ which can be written as

$$\boxed{\frac{d^2 a}{dz^2} + m^2 a = 0}$$

where "m" can be interpreted as a vertical wavenumber

$$m^2 = -\left(\frac{N}{f_0}\right)^2\left(k^2 + l^2 + \frac{\beta_0 k}{\omega}\right)$$

for a solution

$$a(z) = A\cos(mz) + B\sin(mz)$$

$$\frac{da}{dz} = -Am\sin(mz) + Bm\cos(mz)$$

$$\frac{d^2a}{dz^2} = -Am^2\cos(mz) - Bm^2\sin(mz)$$

or $$\frac{d^2 a}{dz^2} = -m^2 \left(A \cos mz + B \sin mz \right)$$
$$= -m^2 \cdot a$$

$\Rightarrow$
$$\frac{d^2 a}{dz^2} + m^2 \cdot a = 0$$

Next we will need vertical boundary conditions to determine the two constants A and B, but we can already find a <u>dispersion relation</u> from

$$m^2 = -\left(\frac{N}{f_0}\right)^2 \left(k^2 + l^2 + \frac{\beta_0 k}{\omega} \right)$$

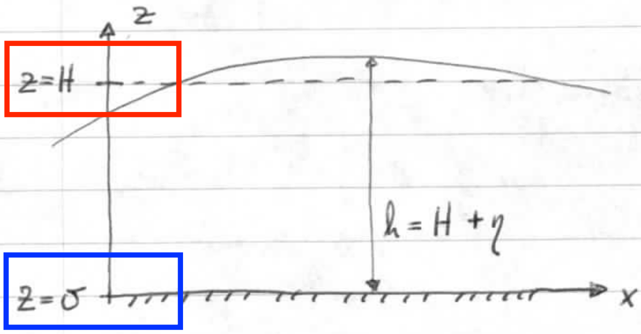
as

$$\boxed{\omega = -\frac{\beta_0 \cdot k}{k^2 + l^2 + (f_0/N)^2 \cdot m^2}}$$

phase velocity in x: $$c_p^{(x)} = \frac{\omega}{k} = -\frac{\beta_0}{k^2 + l^2 + (f\, m/N)^2} < 0$$

always to the West !

Boundary Conditions in "z"



surface @ $z = H + \eta$

bottom @ $z = 0$

Recall $p_0 = \rho_0 f_0 \psi$ Recall $\rho' = -\frac{1}{g}\frac{\partial p_0}{\partial z} = -\frac{\rho_0 f_0}{g}\frac{\partial \psi}{\partial z}$

$$\boxed{(1)\ @\ z = 0\ (\text{bottom})\ :\ w = 0} = \frac{g}{N^2 \rho_0}\left[-\frac{1}{g}\frac{\partial}{\partial t}\left(\frac{\partial p_0}{\partial z}\right) + \underbrace{\cancel{\frac{1}{\rho_0 f_0} J\left(p_0, -\frac{1}{g}\frac{\partial p_0}{\partial z}\right)}}_{\text{"small" non-linear term}}\right]$$

$\to 0$

from density equation (see page 116A)

$1 \gg Ro_T \gg Ro$

$$\Downarrow\ \frac{\partial}{\partial t}\left(\frac{\partial p_0}{\partial z}\right) = 0$$

$$\text{or}\quad \frac{\partial}{\partial t}\left(\rho_0 f_0 \frac{\partial \psi}{\partial z}\right) = 0 \qquad \text{or}\quad \boxed{\frac{\partial \psi}{\partial z} = \boxed{\frac{da}{dz} = \text{const} = 0\,\Big|_{z=0}}}$$

Recall $a(z) = A\cos(mz) + B\sin(mz)$

$$\frac{da}{dz} = -mA\sin(mz) + mB\cos(mz)$$

$$@\ z = 0:\quad \frac{da}{dz} = 0 + mB \cdot 1 \qquad \Downarrow\ B = 0$$

(2) $\boxed{@\ z = h \quad (\text{surface}): \qquad w = \frac{\partial \eta}{\partial t}}$

$$p_o = \rho_o g \eta = \rho_o g \psi \big|_{z=h}$$

Thus $\frac{\partial \eta}{\partial t} = \frac{1}{\rho_o g} \frac{\partial p_o}{\partial t} = \frac{\partial \psi}{\partial t}$

and w as before (from linearized density equation, page 116A)

$$w = \frac{\cancel{g}}{N^2 \rho_o} \cdot (-) \frac{1}{\cancel{g}} \frac{\partial}{\partial t}\left(\frac{\partial p_o}{\partial z}\right) = -\frac{1}{\rho_o N^2} \frac{\partial}{\partial t}\left(\rho_o g \frac{\partial \psi}{\partial z}\right)$$

$\therefore$ $w = \frac{\partial \eta}{\partial t}$ becomes

$$-\frac{g}{N^2} \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial z}\right) = \frac{\partial \psi}{\partial t} \qquad @ \text{ surface } z = h = H + \eta$$

or

$$\frac{\partial}{\partial t}\left(\psi + \frac{g}{N^2} \frac{\partial \psi}{\partial z}\right) = 0 \qquad @ \text{ surface } z = h \approx H$$

or $\frac{\partial \psi}{\partial z} + \frac{N^2}{g} \psi = \text{const} = 0$

$\therefore$ $\boxed{\frac{da}{dz} + \frac{N^2}{g} a = 0 \qquad @ \text{ surface } z \approx H}$