Surface Boundary Condition:

d(a)/dz + N^2/g * a = 0 @ surface z=H

Recall that $a(z) = A \cos(m z)$

and $a(z=H) = A \cos(m \cdot H)$

and $\left.\frac{da}{dz}\right|_{z=H} = - A \cdot m \sin(m \cdot H)$

These solutions for $z = H$ must satisfy the boundary condition at the surface, that is, satisfy

$$\boxed{\frac{da}{dz} + \frac{N^2}{g} a = 0 \qquad \text{at } z = H}$$

$$- A m \sin(m H) + \frac{N^2}{g} \cdot A \cos(m H) = 0$$

or $$- m \cdot \sin(m H) + \frac{N^2}{g} \cos(m H) = 0$$

or $$\frac{\sin(m H)}{\cos(m H)} = + \frac{N^2}{g} \cdot \frac{1}{m}$$

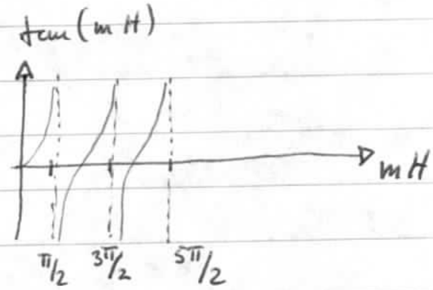
$$\boxed{\tan(m H) = \frac{N^2 \cdot H}{g} \cdot \frac{1}{m H}}$$

Only those "m" that satisfies this "transzendental" equations are allowed to [illegible]

$$\boxed{\tan(mH) = \frac{N^2 H}{g} \cdot \frac{1}{(mH)}}$$

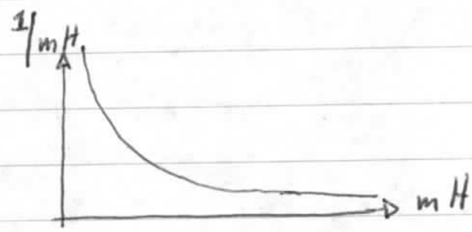
Surface Boundary Condition

This is a transcendental equation that must be solved "graphically":



The left-hand side is a $\tan(m \cdot H)$

The right-hand side is a $1/(m \cdot H)$:



Hence there is a discrete "set" of vertical wave numbers "m" that satisfy both the vorticity equation and the surface boundary conditions.

Show code qg.f

and plot qg.plt

to visualize this operation to find "m"

There is exactly 1 solution for each $(m_i H)$

in each interval $[-\pi/2, +\pi/2] \cdot i \qquad i = 0, 1, 2, \ldots$

$\boxed{\text{For } \underline{i = 0}}$

$$\tan(mH) \approx mH$$

$$\Downarrow \qquad mH \approx \frac{N^2}{g}\frac{H}{H}\frac{1}{m}$$

$$\Downarrow \qquad m^2 \approx \frac{N^2}{gH}$$

And the Dispersion

$$\omega = -\beta k \Big/ \left[k^2 + l^2 + m^2 f_0^2/N^2\right]$$

becomes

$$\omega_0 = -\beta k \Big/ k^2 + l^2 + (1/L_0)^2 \qquad L_0 = \sqrt{gH}\Big/ f_0$$

external (barotropic) Rossby radius

Note that this dispersion does NOT include the stability frequency N that describes the background stratification $\frac{1}{\rho_0}\frac{\partial \bar{\rho}}{\partial z}$

We discussed this exact wave Mar.-13 (page-60) for barotropic dynamics (no stratification or vertical variability

$\boxed{\text{For } \underline{i > 0}}$

$$\omega_i \approx -\beta k \Big/ k^2 + l^2 + \left(i \cdot \pi / L_D\right)^2$$

$$L_D = N \cdot H / f_0$$

$$m_i \approx i \cdot \pi$$

refer to qg.plt

with $N = 0.01 \ s^{-1}$, $H = 1000 \ m$, $f_0 = 10^{-4} \ s^{-1}$ $\Big\} \ \dfrac{N \cdot H}{f_0} = 100 \ km = L_D$

Mode-1 $\dfrac{L_D}{1 \cdot \pi} \approx 30 \ km$

Mode-2 $\dfrac{L_D}{2 \cdot \pi} \approx 15 \ km$

Mode-3 $\dfrac{L_D}{3 \cdot \pi} \approx 10 \ km$

Mode-0 $\dfrac{\sqrt{gH}}{f_0} \approx \dfrac{100 \ m/s}{10^{-4} \ s^{-1}} = 10^6 \ m = 1000 \ km$

Discuss Solutions and Modes graphically

1. z.plt gives $\Psi_0(z)$ or $a(z) = \cos(mz)$

this is a vertical structure function that satisfies surface and bottom boundary conditions
$w = 0$ at bottom $w = \frac{\partial \eta}{\partial t}$ at surface

Frozen in time — __no__ vertical oscillation or vertical phase propagation

2. qg-dispersion 3. plt $\omega = \omega(k \cdot L_D)$ for mode-0 $(i=0)$
mode-1 $(i=1)$
mode-2 $(i=2)$

in cycles per year

refer to page-64

0. qg.plt Finding m_i from $\tan(mH) = \frac{N^2 H}{g} \cdot \frac{1}{mH}$

(surface boundary condition)

Close with the note that a sloping bottom has the

same effect on vorticity dynamics as the planetary β-effect

↑ moving a fluid column north-ward (increase its $f = f_0 + \beta y$)

has similar effects as moving a fluid column into shallower water
(decrease $H = H_0 + \alpha y$)