1.4 Group Velocity

velocity of a wave packet constructed by superposition

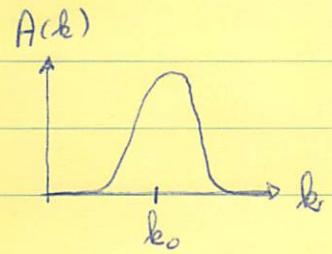
Assume we have some

$$\sigma = \sigma(k) \qquad \text{dispersion relation}$$

Superposition means Sum-up all waves

$$\phi(x,t) = \int \underbrace{A(k)}_{\text{amplitudes}} \; \underbrace{e^{i[kx - \sigma(k)t]}}_{\text{phase } \varphi = kx - \sigma t} \, dk$$

wave packet:

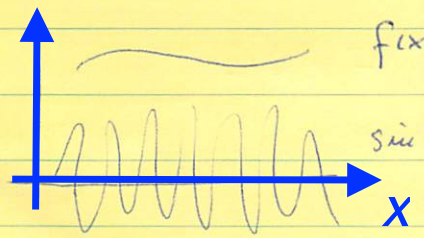


Wave Spectra:
distribution of amplitude
or variance or kinetic energy
for many different waves
with different k

How does this wave packet move?

Method of Stationary Phase

If you have an integral of a rapidly oscillating and a slowly varying function, then you get something close to zero:



$$\int f(x) \sin(x)\, dx \approx 0$$

because the "+" cancels the "−"

↳ need slowly varying phase to get contributions to the integral near k_0

So we need

$$\varphi(k) = kx - \sigma(k)\cdot t$$

Phase of Wave

to become "stationary", that is, to very very little:

$$\frac{d\varphi}{dk} = x - \left.\frac{d\sigma}{dk}\right|_{k_0} \cdot t = 0 \;?$$

$$\Rightarrow \quad x = \boxed{\left.\frac{d\sigma}{dk}\right|_{k_0}} \cdot t$$

group velocity $\quad c_g\big|_{k_0} = x/t$

Next we need to show that the wave pattern $\phi(x,t)$ indeed moves at this speed. For this we need to solve the integral

#3

$$\phi(x, t=0) = \int A(k)\, e^{ikx}\, dk$$

#0 #1,#2,#3

$$\phi(x,t) = \int A(k)\, e^{i(kx - \sigma t)}\, dk$$

All we know is that our wave packet peaks near k_0

$$\Rightarrow \quad \sigma(k) = \sigma(k_0) + (k-k_0)\left.\frac{d\sigma}{dk}\right|_{k_0} + O\left((k-k_0)^2\right)$$

Taylor Expansion #1 #2 #3

So

$$\phi(x,t) = \int A(k)\, e^{ikx}\, \underbrace{e^{-i\sigma(k_0)t}}_{\#0}\; e^{\overset{\#1}{-ik\frac{d\sigma}{dk}\Big|_{k_0}t}}\; \underbrace{e^{\overset{\#3}{+ik_0\frac{d\sigma}{dk}\Big|_{k_0}t}}}_{\text{negligible}}\, dk$$

$$\phi(x,t) = \underbrace{e^{-it\left[\overset{\#1}{\sigma(k_0)} - \overset{\#3}{\frac{d\sigma}{dk}\Big|_{k_0}}\right]}}_{\text{this is just a pure phase whose magnitude is always } \underline{1}} \cdot \underbrace{\int A(k)\, e^{\overset{\#0}{ik}\left[x - \overset{\#2}{\frac{d\sigma}{dk}\Big|_{k_0}}\cdot t\right]}\, dk}_{\text{this "looks" like } \phi(x', t=0) \text{ with}}$$

exp[-i * sigma * t
or cos[sigma*t]

$$x' = x - \frac{d\sigma}{dk}\Big|_{k_0} \cdot t$$

$$|\phi(x,t)| = \underline{1} \cdot \phi\left(x - \frac{d\sigma}{dk}\Big|_{k_0} \cdot t,\ 0\right)$$

The wave packet @ time t is the same as at time $t=0$

but move ~~to~~ from x to $\left(x - \frac{d\sigma}{dk}\Big|_{k_0} \cdot t\right)$

This defines the so-called "group velocity" of a wave

as the derivative of frequency with regard to wave number.

Recall that the phase velocity of a wave was defined

as the ratio of frequency and wave number.