PART IV

COMBINED ROTATION AND STRATIFICATION EFFECTS

12

Layered Models

Summary: Advantage is taken of the assumption of density conservation by fluid parcels to change the vertical coordinate from depth to density. The new equations allow for a clear discussion of potential-vorticity dynamics and lend themselves to discretization in the vertical. The result is a layered model. Note: To avoid problems of terminology, we restrict ourselves here to the ocean. The case of the atmosphere follows with the replacement of depth by height and density by potential density.

12-1 FROM DEPTH TO DENSITY

Since a stable stratification requires a monotonic increase of density downward, density can be taken as a surrogate for depth and used as the vertical coordinate. If density is conserved by individual fluid parcels, as it is approximately the case for most geophysical flows, considerable mathematical simplification follows, and the new equations present a definite advantage in a number of situations. It is thus worth expounding on this change of variables at some length.

In the original Cartesian system of coordinates, z is an independent variable and density $\rho(x, y, z, t)$ is a dependent variable, giving the water density at location (x, y), time t, and depth z. In the transformed coordinate system (x, y, ρ, t), density becomes an independent variable and $z(x, y, \rho, t)$ has become the dependent variable, giving the depth at which density ρ is found at location (x, y) and at time t.

From a differentiation of the expression $a = a(x, y, \rho(x, y, z, t), t)$, where a is any variable, the rules for the change of variables follow:

$$\frac{\partial}{\partial x} \to \left.\frac{\partial a}{\partial x}\right|_z = \left.\frac{\partial a}{\partial x}\right|_\rho + \frac{\partial a}{\partial \rho} \left.\frac{\partial \rho}{\partial x}\right|_z$$

$$\frac{\partial}{\partial y} \to \left.\frac{\partial a}{\partial y}\right|_z = \left.\frac{\partial a}{\partial y}\right|_\rho + \frac{\partial a}{\partial \rho} \left.\frac{\partial \rho}{\partial y}\right|_z$$

$$\frac{\partial}{\partial z} \to \frac{\partial a}{\partial z} = \frac{\partial a}{\partial \rho} \frac{\partial \rho}{\partial z}$$

$$\frac{\partial}{\partial t} \to \left.\frac{\partial a}{\partial t}\right|_z = \left.\frac{\partial a}{\partial t}\right|_\rho + \frac{\partial a}{\partial \rho} \left.\frac{\partial \rho}{\partial t}\right|_z .$$

Then, the application of $a = z$ allows the change of derivatives of ρ at z constant to those of z at ρ constant and to write

$$\left.\frac{\partial a}{\partial x}\right|_z = \left.\frac{\partial a}{\partial x}\right|_\rho - \frac{z_x}{z_\rho} \frac{\partial a}{\partial \rho}, \qquad (12\text{-}1)$$

with similar expressions where x is replaced by y or t, and

$$\frac{\partial a}{\partial z} = \frac{1}{z_\rho} \frac{\partial a}{\partial \rho}. \qquad (12\text{-}2)$$

Here, subscripts denote derivatives. Figure 12-1 depicts a geometrical interpretation of rule (12-1).

The hydrostatic equation (3-27) readily becomes

$$\frac{\partial p}{\partial \rho} = -\rho g \frac{\partial z}{\partial \rho} \qquad (12\text{-}3)$$

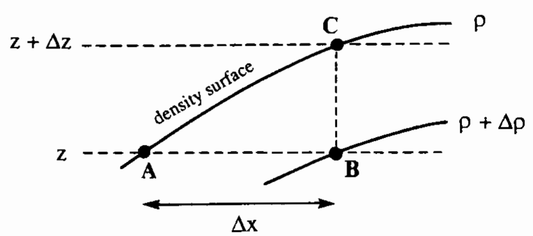

Figure 12-1 Geometrical interpretation of equation (12-1). The x-derivatives of any function a at constant depth and at constant density are, respectively, $[a(B) - a(A)]/\Delta x$ and $[a(C) - a(A)]/\Delta x$. The difference between the two, $[a(C) - a(B)]/\Delta x$, represents the vertical derivative of a, $[a(C) - a(B)]/\Delta z$, times the slope of the density surface, $\Delta z/\Delta x$. Finally, the vertical derivative can be split as the ratio of the ρ-derivative of a, $[a(C) - a(B)]/\Delta\rho$, by $\Delta z/\Delta\rho$.

and leads to the following horizontal pressure gradient:

$$\left.\frac{\partial p}{\partial x}\right|_z = \left.\frac{\partial p}{\partial x}\right|_\rho - \frac{z_x}{z_\rho}\frac{\partial p}{\partial \rho} = \left.\frac{\partial p}{\partial x}\right|_\rho + \rho g \frac{\partial z}{\partial x} = \left.\frac{\partial P}{\partial x}\right|_\rho .$$

Similarly, $\partial p/\partial y$ at constant z becomes $\partial P/\partial y$ at constant ρ. The new function P, which plays the role of pressure in the density-coordinate system, is defined as

$$P = p + \rho g z \tag{12-4}$$

and is called the *Montgomery potential*, in honor of Raymond B. Montgomery, who introduced it for the first time in 1937. Later on, when there is no ambiguity, this potential may loosely be called pressure. With P replacing pressure, the hydrostatic balance, (12-3), now takes a more compact form:

$$\frac{\partial P}{\partial \rho} = g z, \tag{12-5}$$

further indicating that P is the natural substitute for pressure when density is the vertical coordinate. Beyond this point, all derivatives with respect to x, y, and time are meant to be taken at constant density, and the subscript ρ is no longer necessary.

In the absence of diffusion, the density-conservation equation, (3-29), can be solved for the vertical velocity

$$w = \frac{\partial z}{\partial t} + u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y}. \tag{12-6}$$

This last equation simply tells that the vertical velocity is that necessary for the particle to remain at all times on the same density surface. Armed with expression (12-6), we can now eliminate the vertical velocity throughout the set of governing equations. First, the material derivative (3-4) assumes a simplified, two-dimensional-like form:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \tag{12-7}$$

where the derivatives are now taken at constant ρ. The absence of a third, verticallike advective term physically results from the assumed absence of motion across density surfaces.

In the absence of friction, the horizontal-momentum equations (3-25) and (3-26) become

$$\frac{du}{dt} - fv = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} \tag{12-8}$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho_0}\frac{\partial P}{\partial y}. \tag{12-9}$$

We note that they are almost identical to their original versions. The differences are nonetheless important: The material derivative has been reduced to (12-7), the pressure has been replaced by the Montgomery potential P defined in (12-4), and all temporal and horizontal derivatives are taken at constant density. Note, however, that the components u and v are still the horizontal velocity components and are not measured along sloping density surfaces. This property is important for the proper application of lateral boundary conditions.

To complete the set of equations, it remains to transform the continuity equation, (3-28), according to rules (12-1) and (12-2). Further elimination of the vertical velocity by using of (12-6) leads to

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0, \qquad (12\text{-}10)$$

where the quantity h is introduced for convenience and is proportional to $\partial z/\partial \rho$, the derivative of depth with respect to density. For convenience, we want h to have the dimension of height, so we introduce an arbitrary but constant density difference, $\Delta\rho$, and define

$$h = -\Delta\rho \frac{\partial z}{\partial \rho}. \qquad (12\text{-}11)$$

In this manner, h can be interpreted as the thickness of a fluid layer between the density ρ and $\rho + \Delta\rho$. At this point, the value of $\Delta\rho$ is totally arbitrary, but later, in the development of layered models, it will be chosen as the density difference between adjacent layers.

The transformation of coordinates is now complete. The new set of governing equations consists of the two horizontal-momentum equations (12-8) and (12-9), the hydrostatic balance (12-5), the continuity equation (12-10), and the relation (12-11). It thus forms a closed 5-by-5 system for the dependent variables, u, v, P, z, and h. Once the solution is known, the pressure p and the vertical velocity w can be recovered from (12-4) and (12-6).

Since the aforementioned work of Montgomery (1937), the substitution of density as the vertical variable has been implemented in a number of applications, especially by Robinson (1965) in a study of inertial currents, by Hodnett (1978) in a study of the permanent oceanic thermocline, and by Sutyrin (1989) in a study of isolated eddies. A review in the meteorological context is provided by Hoskins et al. (1985).

12-2 POTENTIAL VORTICITY

Potential vorticity is a dynamic quantity conserved by individual fluid parcels in inviscid, nondiffusive flows. Our earlier study of homogeneous rotating fluids (Section 4-4) led to an expression for potential vorticity, which was then interpreted as circulation per volume. We will now show that the same expression is applicable to stratified rotating fluids if the quantity h is no longer the fluid depth but is the variable defined in (12-11). The interpretation of potential vorticity as circulation per volume and the conservation principle remain unchanged. The following presentation is a variant on the original derivation by Rossby (1940).

A cross-differentiation of the horizontal-momentum equations (12-8) and (12-9) eliminates the pressure and yields

$$\frac{d}{dt}\left(f+\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\left(f+\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0, \qquad (12\text{-}12)$$

while the continuity equation (12-10) can be recast into a similar form:

$$\frac{dh}{dt}+h\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0. \qquad (12\text{-}13)$$

Consider now a volume element of cross-section ds sandwiched between the density surfaces ρ and $\rho + \Delta\rho$. It is a straightforward geometric result that, due to lateral divergence or convergence, the parcel's horizontal cross-section ds will expand or shrink according to (see Section 4-4)

$$\frac{d}{dt}\,ds=\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)ds. \qquad (12\text{-}14)$$

Combining (12-14) first with (12-12) and then with (12-13) yields the following conservation relations:

$$\frac{d}{dt}\left[\left(f+\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)ds\right]=0, \qquad (12\text{-}15)$$

$$\frac{d}{dt}\,(h\;ds)=0. \qquad (12\text{-}16)$$

The former states that the circulation within the parcel and about the vertical axis is conserved, whereas the latter states that the volume of fluid within the element remains unchanged. Consequently, a fluid parcel that is shrinking laterally (i.e., decreasing ds) will become taller (increasing h) and acquire a greater vorticity (increasing $f + \partial v/\partial x - \partial u/\partial y$). Similarly, a lateral expansion is accompanied by a reduction in thickness and vorticity. The scenario is illustrated in Figure 12-2.

Now, if both circulation and volume are conserved, so must be their ratio; that is,

$$\frac{dq}{dt}=0, \qquad (12\text{-}17)$$

where

$$q=\frac{f+\partial v/\partial x-\partial u/\partial y}{h}. \qquad (12\text{-}18)$$

This quantity, called the potential vorticity [see (4-27)], can be thought as the circulation per volume. It is conserved because both circulation and volume are conserved. In most applications, potential vorticity takes precedence over both circulation and

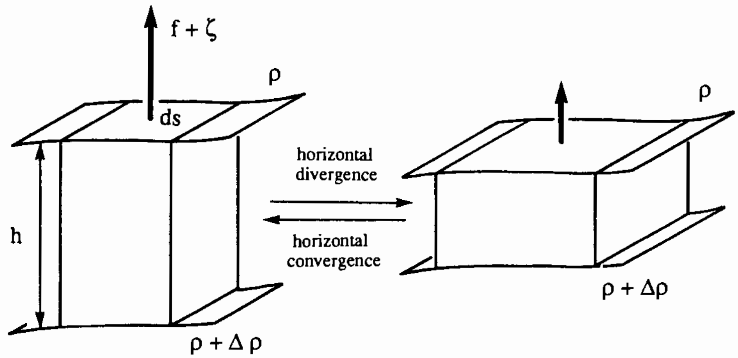

Figure 12-2 Conservation of volume and circulation in a fluid parcel undergoing divergence (squeezing) or convergence (stretching). The products $h\ ds$ and $(f + \zeta)\ ds$ are conserved during the transformation.

volume because its evaluation does not involve the fluid parcel's lateral cross-section. Recall that in the present derivation, the x- and y-derivatives are not taken horizontally but along the sloping density surfaces and that the variable h is not a true depth but a quantity defined as proportional to the derivative of depth with respect to density.

In closing this section, it is worth noting that the expression of potential vorticity in the original Cartesian coordinates is much more complicated. We will simply state its expression without demonstration:

$$q = \left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \frac{\partial \rho}{\partial z} + \left(\frac{\partial u}{\partial z}\frac{\partial \rho}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial \rho}{\partial x}\right). \tag{12-19}$$

12-3 LAYERED MODELS

A *layered model* is an ideal fluid system that consists of a finite number of moving layers, stacked one upon another and each having a uniform density. Its evolution is governed by a discretized version of the system of equations in which density, taken as the vertical variable, is not varied continuously but is restricted to assume a finite number of values. A layered model is the density analogue of a *level model*, which is obtained after discretization of the vertical variable z.

Each layer ($i = 1$ to n, where n is the number of layers) is described by its density ρ_i (unchanging), thickness h_i, Montgomery potential P_i, and horizontal velocity components u_i and v_i. The surface marking the boundary between two adjacent layers is called an *interface* and is described by its elevation z_i, measured (negatively downward) from the mean surface level. The displaced surface level is called z_0 (Figure 12-3a). The interfacial heights can be obtained recursively from the bottom,

$$z_n = -H, \tag{12-20a}$$

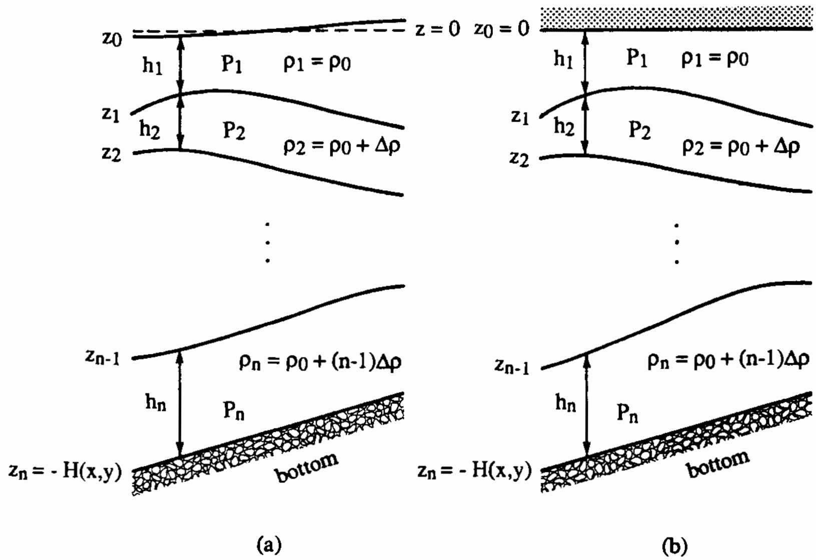

Figure 12-3 A layered model with n active layers: (a) with free surface, (b) with rigid lid.

upward:

$$z_{i-1} = z_i + h_i, \quad i = n \text{ to } 1. \tag{12-20b}$$

This geometrical relation can be regarded as the discretized version of (12-11) used to define h.

In a similar manner, the discretization of hydrostatic relation (12-5) provides another recursive relation, which can be used to evaluate the Montgomery potential P from the top,

$$P_1 = p_a + \rho_0 g z_0, \tag{12-21a}$$

downward:

$$P_{i+1} = P_i + \Delta\rho g z_i, \quad i = 1 \text{ to } n-1. \tag{12-21b}$$

To write (12-21a), we have selected the uppermost density ρ_1 as the reference density ρ_0. Gradients of the atmospheric pressure p_a typically play no significant role, and the contribution of p_a to P_1 may be omitted. If the layered model is for the lower atmosphere, p_a represents a pressure distribution aloft and may again be taken as an inactive constant and thus dropped.

When the *reduced gravity*,

$$g' = \frac{\Delta\rho}{\rho_0} g, \tag{12-22}$$

is introduced for convenience, the recursive relations (12-20b) and (12-21b) lead to simple expressions for the interfacial heights and Montgomery potentials. For up to three layers, these are as follows:

One layer: (12-23)

$z_0 = h_1 - H \qquad P_1 = \rho_0 g(h_1 - H)$

$z_1 = -H$

Two layers: (12-24)

$z_0 = h_1 + h_2 - H \qquad P_1 = \rho_0 g(h_1 + h_2 - H)$

$z_1 = h_2 - H \qquad P_2 = \rho_0 g h_1 + \rho_0(g + g')(h_2 - H)$

$z_2 = -H$

Three layers: (12-25)

$z_0 = h_1 + h_2 + h_3 - H \qquad P_1 = \rho_0 g(h_1 + h_2 + h_3 - H)$

$z_1 = h_2 + h_3 - H \qquad P_2 = \rho_0 g h_1 + \rho_0(g + g')(h_2 + h_3 - H)$

$z_2 = h_3 - H \qquad P_3 = \rho_0 g h_1 + \rho_0(g + g')h_2 + \rho_0(g + 2g')(h_3 - H)$

$z_3 = -H.$

In certain applications, it is helpful to discard the surface gravity waves, which travel much faster than internal waves and near-geostrophic disturbances. To do so, we eliminate the flexibility of the surface by imagining that the system is covered by a rigid lid (Figure 12-3b). This is called the *rigid-lid approximation*. In such a case, z_0 is equal to zero, and there are only $(n - 1)$ independent layer thicknesses. In return, one of the Montgomery potentials cannot be derived from the hydrostatic relation. If this potential is chosen as the one in the lowest layer, the recursive relations yield the following:

One layer: (12-26)

$z_1 = -h_1 \qquad P_1$ variable

$h_1 = H$, fixed

Two layers: (12-27)

$z_1 = -h_1 \qquad P_1 = P_2 + \rho_0 g' h_1$

$z_2 = -h_1 - h_2 \qquad P_2 =$ variable

$h_1 + h_2 = H$, fixed

Three layers: (12-28)

$z_1 = -h_1 \qquad P_1 = P_3 + \rho_0 g'(2h_1 + h_2)$

$z_2 = -h_1 - h_2 \qquad P_2 = P_3 + \rho_0 g'(h_1 + h_2)$

$z_3 = -h_1 - h_2 - h_3 \qquad P_3 = \text{variable}$

$h_1 + h_2 + h_3 = H$, fixed.

In some other instances, mainly in the investigations of upper-ocean processes, the lowest layer can be imagined to be infinitely deep and at rest. Keeping n as the number of moving layers, we assign to this lowest (abyssal) layer the index $(n + 1)$. The absence of motions there implies a uniform Montgomery potential, the value of which can be set to zero without loss of generality: $P_{n+1} = 0$. For up to three active layers, the recursive relations provide (Figure 12-4) the following:

One layer: (12-29)

$$z_1 = -h_1 \qquad P_1 = \rho_0 g' h_1$$

Two layers: (12-30)

$$z_1 = -h_1 \qquad P_1 = \rho_0 g'(2h_1 + h_2)$$

$$z_2 = -h_1 - h_2 \qquad P_2 = \rho_0 g'(h_1 + h_2)$$

Three layers: (12-31)

$$z_1 = -h_1 \qquad P_1 = \rho_0 g'(3h_1 + 2h_2 + h_3)$$

$$z_2 = -h_1 - h_2 \qquad P_2 = \rho_0 g'(2h_1 + 2h_2 + h_3)$$

$$z_3 = -h_1 - h_2 - h_3 \qquad P_3 = \rho_0 g'(h_1 + h_2 + h_3).$$

Because these expressions do not involve the full gravity g but only its reduced value g', this type of model is known as a *reduced-gravity model.*

Generalization to more than three moving layers is straightforward. When a configuration with few but physically relevant layers is desired, the preceding derivations may be extended to nonuniform density differences from layer to layer. Mathematically,

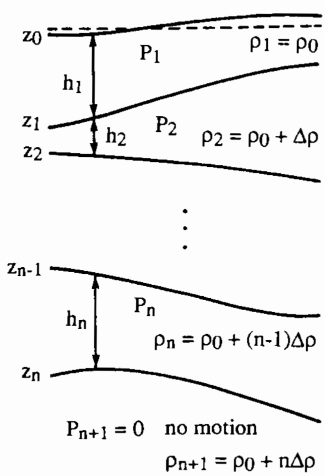

Figure 12-4 A reduced-gravity layered model.

this would correspond to a discretization of the vertical density axis into unevenly spaced gridpoints.

Once the layer thicknesses, interface depths, and layer pressures (more precisely, the Montgomery potentials) are all related, the system of governing equations is completed by gathering the horizontal-momentum equations (12-8) and (12-9) and the continuity equation (12-10), each written for every layer. The expression for potential vorticity, (12-18), is unchanged, except that the denominator is now the finite thickness of the layer for which it is constructed.

In Section 9-5, the length $L = NH/\Omega$ was derived as the horizontal scale at which rotation and stratification play equally important roles. It is noteworthy at this point to formulate the analogue for a layered system. Introducing H as a typical layer thickness in the system (such as the maximum depth of the uppermost layer at some initial time) and $\Delta\rho$ as a density difference between two adjacent layers (such as the top two), an approximate expression of the stratification frequency squared is

$$N^2 = -\frac{g}{\rho_0}\frac{d\rho}{dz} \simeq \frac{g}{\rho_0}\frac{\Delta\rho}{H} = \frac{g'}{H}, \tag{12-32}$$

where $g' = g\ \Delta\rho/\rho_0$ is the reduced gravity defined earlier. Substitution of (12-32) in the definition of L yields $L \simeq (g'H)^{1/2}/\Omega$. Finally, because the ambient rotation rate Ω enters the dynamics only via the Coriolis parameter f, it is more convenient to introduce the length scale

$$R = \frac{\sqrt{g'H}}{f}, \tag{12-33}$$

called the radius of deformation. To distinguish this last scale from its cousin (6-10), derived for free-surface homogeneous rotating fluids (where the full gravitational acceleration g appears), it is customary in situations where ambiguity could arise to use the expressions *internal radius of deformation* and *external radius of deformation* for (12-33) and (6-10), respectively. Because density differences within geophysical fluids are typically a percent or less of the average density, the internal radius is one-tenth the external radius and usually less.

When the model consists of a single moving layer above a motionless abyss, the governing equations reduce to

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g'\frac{\partial h}{\partial x}, \tag{12-34}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g'\frac{\partial h}{\partial y}, \tag{12-35}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0. \tag{12-36}$$

The subscripts indicating the layer have become superfluous and have been deleted. The coefficient $g' = g(\rho_2 - \rho_1)/\rho_0$ is called the reduced gravity. Except for the replacement of the full gravitational acceleration, g, by its reduced fraction, g', this system of equations is identical to that of the shallow-water model (Section 4-3) and is thus called the *shallow-water reduced-gravity model.* Because the vertical simplicity of this model permits the investigation of a number of horizontal processes with a minimum of mathematical complications, it will be used in the following chapters. Finally, recall that the Coriolis parameter, f, can be taken as either a constant (f-plane) or as a function of latitude ($f = f_0 + \beta_0 y$, beta plane).

PROBLEMS

12-1. Generalize the theory of the coastal Kelvin wave (Section 6-2) to the two-layer system over a flat bottom and under a rigid lid. In particular, what are the wave speed and trapping scale?

12-2. In the case of the shallow-water reduced-gravity model, derive an energy-conservation principle. Then, separate the kinetic and potential energy contributions.

12-3. Show that a steady flow of the shallow-water reduced-gravity system conserves the Bernoulli function $B = g'h + (u^2 + v^2)/2$.

12-4. Establish the equations governing motions in a one-layer model above an uneven bottom and below a thick, motionless layer of slightly lesser density.

12-5. Seek a solution to the shallow-water reduced-gravity model of the type $h(x, t) = A(t)x^2 + 2B(t)x + C(t)$, $u(x, t) = U_1(t)x + U_0(t)$, $v(x, t) = V_1(t)x + V_0(t)$. To what type of motion does this solution correspond? What can you say of its temporal variability? (Take f = constant.)

Raymond Braislin Montgomery

1910 – 1988

A student of Carl-Gustav Rossby, Raymond Braislin Montgomery earned his fame as a brilliant descriptive physical oceanographer. Applying dynamic results derived by his mentor and other contemporary theoreticians to observations, he developed precise means of characterizing water masses and currents. By his choice of analyzing observations along density surfaces rather than along level surfaces, an approach that led him to formulate the potential now bearing his name, Montgomery was able to trace the flow of water masses across ocean basins and to arrive at a lucid picture of the general oceanic circulation. Montgomery's lectures and published works, marked by an unusual attention to clarity and accuracy, earned him great respect as a critic and reviewer. (*Photo by Hideo Akamatsu; courtesy of Mrs. R. B. Montgomery.*)