13

Stratified Geostrophic Dynamics

Summary: This chapter shows that geostrophic motions can maintain a stratified fluid away from gravitational equilibrium. The key is a relationship between the horizontal density gradient and the vertical velocity shear, called the thermal-wind relation. How such steady flows can originate is illustrated by the investigation of geostrophic adjustment, the process by which a stratified fluid out of equilibrium tends to assume a steady, geostrophic state.

13-1 THERMAL WIND

Consider a situation where a cold air mass is wedged between the ground and a warm air mass (Figure 13-1). The stratification has then both vertical and horizontal components. Mathematically, the density is a function of both z (height) and x (from cold to warm). Now, assume that the flow is steady, geostrophic, and hydrostatic:

$$- f v = - \frac{1}{\rho_0} \frac{\partial p}{\partial x} \tag{13-1}$$

$$\frac{\partial p}{\partial z} = - \rho g . \tag{13-2}$$

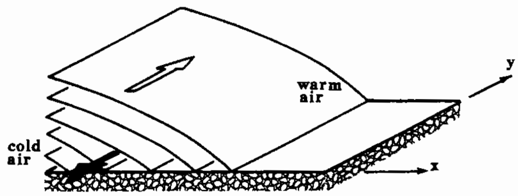


Figure 13-1 Vertical velocity shear in the presence of a horizontal density gradient. The change of velocity with height is called the thermal wind.

Here, v is the velocity component in the horizontal direction y, and p is the pressure field. Taking the z-derivative of (13-1) and eliminating $\partial p/\partial z$ with (13-2), we obtain

$$\frac{\partial v}{\partial z} = -\frac{g}{\rho_0 f}\frac{\partial \rho}{\partial x}. \tag{13-3}$$

Therefore, a horizontal density gradient can exist in a steady state if it is accompanied by a vertical shear of horizontal velocity. Where density varies in both horizontal directions, the following also holds:

$$\frac{\partial u}{\partial z} = +\frac{g}{\rho_0 f}\frac{\partial \rho}{\partial y}. \tag{13-4}$$

These innocent-looking relations have profound meaning. They state that, due to the Coriolis force, the system can be maintained in equilibrium, without tendency toward leveling of the density surfaces. In other words, the rotation of the earth can keep the system away from its state of rest without any continuous supply of energy.

Notice that the velocity field (u, v) is not specified, only its vertical shear, $\partial u/\partial z$ and $\partial v/\partial z$. This implies that the velocity must change with height. (In the case of Figure 13-1, $\partial \rho/\partial x$ is negative and $\partial v/\partial z$ is positive.) For example, the wind speed and direction at some height above the ground may be totally different from those at ground level. The presence of a vertical gradient of velocity also implies that the velocity cannot vanish, except perhaps at some discrete levels. Meteorologists have named such a flow the *thermal wind.* Another name is *baroclinic flow.*

In the case of pronounced density contrasts, such as across cold and warm fronts, a layered system may be applicable. In this case (Figure 13-2), the system can be represented by two densities (ρ_1 and ρ_2, $\rho_1 < \rho_2$) and two velocities (v_1 and v_2). Relation (13-3) can be discretized into

$$\frac{\Delta v}{\Delta z} = -\frac{g}{\rho_0 f}\frac{\Delta \rho}{\Delta x},$$

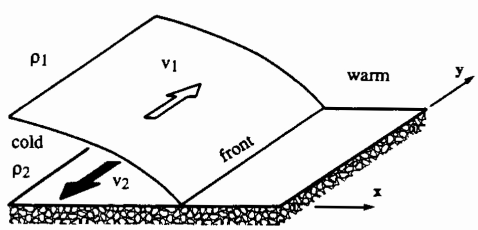


Figure 13-2 The layered version of Figure 13-1, which leads to the Margules relation.

where we take $\Delta v = v_1 - v_2$ and $\Delta\rho = \rho_2 - \rho_1$ to obtain

$$v_1 - v_2 = -\frac{g}{\rho_0 f}(\rho_2 - \rho_1)\frac{\Delta z}{\Delta x}. \tag{13-5}$$

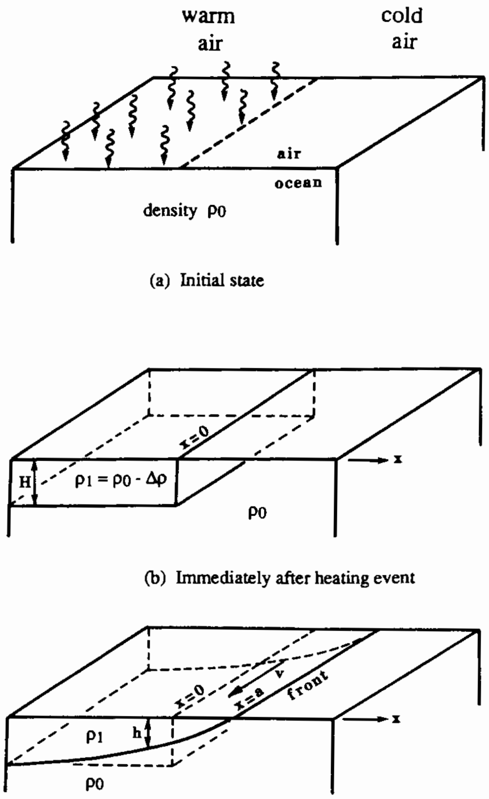
The ratio $\Delta z/\Delta x$ is to be taken as the slope of the interface. This equation is called the *Margules relation* (Margules, 1906), although a more general form of the relation for zonal flows was obtained earlier by Helmholtz (1888).

The thermal-wind concept has been enormously useful in analyzing both atmospheric and oceanic data, because observations of the temperature and other variables that influence the density (such as pressure and moisture in the air, or salinity in the sea water) are typically much more abundant than velocity data. For example, knowledge of temperature and moisture distributions with height and of the surface wind (to start the integration) permits the calculation of wind speed and direction above ground. In the ocean, especially in studies of large-scale oceanic circulation, for which sparse current-meter data may not be representative of the large flow due to local eddy effects, the basinwide distribution may be considered as unknown. For this reason, oceanographers typically assume that the currents vanish at some great depth (e.g., 2000 m) and integrate the "thermal-wind" relations from there upward to estimate the surface currents. Although the method is convenient (the equations are linear and do not require integration in time), we should keep in mind that the thermal-wind relation of (13-3) and (13-4) is rooted in an assumption of strict geostrophic balance. Obviously, this will not be true everywhere and at all times.

13-2 GEOSTROPHIC ADJUSTMENT

We may now ask how situations like the ones depicted in Figures 13-1 and 13-2 can arise. In the atmosphere, latitudinal differences create different air masses, which storms may then redistribute quite rapidly. Storms at sea can also cool the ocean differentially. Finally, coastal processes such as fresh-water runoff can create density differences between offshore waters and those closer to shore. Thus a variety of mechanisms exists by which different fluid masses can be brought in contact. Once contact is established, adjustment follows.

Let us explore the adjustment in its simplest form by imagining an infinitely deep ocean that is suddenly heated over half of its extent (Figure 13-3a). A warm upper layer develops on that side, while the rest of the ocean, on the other side and below, remains relatively cold (Figure 13-3b). (We could also imagine a vertical gate preventing buoyant water from spilling from one side to the other.) After the upper layer has been created—or, equivalently, if the gate is removed—the ocean is not in a state of equilibrium, the lighter surface water spills over, and an adjustment takes place. In the absence of rotation, as we can easily imagine, spilling proceeds until the light water has spread evenly over the entire domain and the system has come to rest. But, this scenario, as we are about to note, is not what happens when rotational effects are important.



(a) Initial state

(b) Immediately after heating event

(c) After ocean's adjustment

Figure 13-3 A simple case of geostrophic adjustment.

Under the influence of the Coriolis force, the forward acceleration induced by the initial spilling creates a current that veers (to the right in the Northern Hemisphere) and can come into geostrophic equilibrium with the pressure difference associated with the density heterogeneity. The result is a partial spill accompanied by a lateral flow (Figure 13-3c).

To model the process mathematically, we use the reduced-gravity model (12-34) through (12-36) on an f-plane and where the reduced-gravity constant is $g' = g(\rho_0 - \rho_1)/\rho_0$ according to the notation of Figure 13-3b. We neglect all variations in the y-direction, although we allow for a velocity, v, in that direction, and write

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} - fv = -g'\,\frac{\partial h}{\partial x} \tag{13-6}$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + fu = 0 \tag{13-7}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0\,. \tag{13-8}$$

The initial conditions (i.e., immediately after the warming event) are $u = v = 0$, $h = H$ for $x < 0$, and $h = 0$ for $x > 0$. The boundary conditions are u, $v \to 0$ and $h \to H$ as $x \to -\infty$, whereas $u = da/dt$, $h = 0$ at $x = a(t)$, the moving point where the interface outcrops. This nonlinear problem cannot be solved analytically, but one property can be stated. The preceding equations conserve the following form of the potential vorticity:

$$q = \frac{f + \partial v/\partial x}{h}.$$

Initially, all particles have $v = 0$, $h = H$ and thus have the same potential vorticity: $q = f/H$. Therefore, throughout the layer of light fluid and at all times, the potential vorticity keeps the uniform value f/H:

$$\frac{f + \partial v/\partial x}{h} = \frac{f}{H}. \tag{13-9}$$

This property, it turns out, allows us to relate the initial state to the final state without having to solve for the complex, intermediate evolution.

Once the adjustment is completed, time derivatives vanish. Equation (13-8) then requires that hu be a constant; since $h = 0$ at one point, this constant must be zero, implying that u must be zero everywhere. Equation (13-7) reduces to zero equals zero and tells nothing. Finally, equation (13-6) implies a geostrophic balance,

$$-fv = -g' \frac{dh}{dx}, \tag{13-10}$$

between the velocity and the pressure gradient set by the sloping interface. Alone, equation (13-10) presents one relation between two unknowns, the velocity and the depth profile. The potential-vorticity conservation principle (13-9), which still holds at the final state, provides the second equation, thereby conveying the information about the initial disturbance into the final state.

Despite the nonlinearities of the original governing equations (13-6) through (13-8), the problem at hand, (13-9) and (13-10), is perfectly linear, and the solution is trivial:

$$h = H\left[1 - \exp\left(\frac{x-a}{R}\right)\right] \tag{13-11}$$

$$v = -\sqrt{g'H}\,\exp\left(\frac{x-a}{R}\right), \tag{13-12}$$

where R is the deformation radius, defined by

$$R = \frac{\sqrt{g'H}}{f}, \tag{13-13}$$

and a is the unknown position of the outcrop (where h vanishes). To determine this distance, we must again tie the initial and final states, this time by imposing volume conservation. Ruling out a finite displacement at infinity where there is no activity, we require that the depletion of light water on the left of $x = 0$ be exactly canceled by the presence of light water on the right; that is,

$$\int_{-\infty}^{0} (H - h)\ dx = \int_{0}^{a} h\ dx, \tag{13-14}$$

which yields

$$a = R = \frac{\sqrt{g'H}}{f}. \tag{13-15}$$

Thus, the maximum distance over which the light water has spilled is none other than the radius of deformation, hence the name of the latter.

Notice that R has the Coriolis parameter f in its denominator. Therefore, the spreading distance, R, is less than infinity because f differs from zero. In other words, the spreading is confined because of the earth's rotation via the Coriolis effect. In a nonrotating framework, the spreading would, of course, be unlimited.

Lateral heterogeneities are constantly imposed onto the atmosphere and oceans, which then adjust and establish patterns whereby these lateral heterogeneities are somewhat distorted but maintained. Such patterns are at or near geostrophic equilibrium and can thus persist for quite a long time. This explains why discontinuities such as fronts are common occurrences in both the atmosphere and the oceans. As the preceding example suggests, fronts and the accompanying winds or currents take place over distances on the order of the deformation radius. To qualify the activity observed at that length scale, meteorologists refer to the *synoptic scale*, whereas oceanographers prefer to use the adjective *mesoscale*.

We can vary the initial, hypothetical disturbance and generate a variety of geostrophic fronts, all being steady states. A series of examples, taken from published studies, is provided in Figure 13-4. They are, in order: surface-to-bottom front on a flat bottom, which can result from sudden and localized heating (or cooling); surface-to-bottom front at the shelf break resulting from the existence of distinct shelf and deep water masses; double, surface-to-surface front; and three-layer front as the result of localized mixing of an otherwise two-layer stratified fluid. The interested reader is referred to the original articles by C. G. Rossby (1937, 1938), the article by Veronis (1956), the review by Blumen (1972), and other articles on specific situations, by Stommel and Veronis (1980), Hsueh and Cushman-Roisin (1983), and van Heijst (1985). Ou (1984) considered the geostrophic adjustment of a continuously stratified fluid and showed that, if the initial situation is sufficiently away from equilibrium, density discontinuities can arise during the adjustment process. In other words, *fronts* can spontaneously emerge from earlier nondiscontinuous conditions.

The preceding applications dealt with situations in which there is no variation in one of the two horizontal directions. The general case (see Hermann et al., 1989) may yield a time-dependent flow that is only nearly geostrophic.

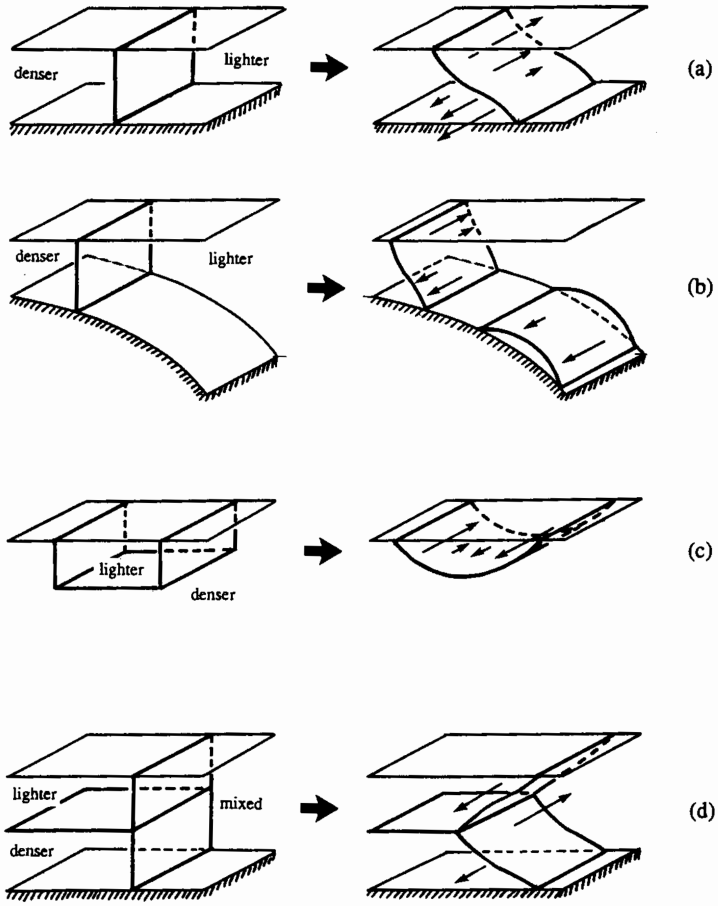


Figure 13-4 Various examples of geostrophic adjustment.

13-3 ENERGETICS OF GEOSTROPHIC ADJUSTMENT

The preceding theory of geostrophic adjustment considered potential-vorticity and volume conservation principles, but nothing was said of energy, which must also be conserved in a nondissipative system. All we can do, now that the solution has been obtained, is to check on the budget, where a surprise is awaiting us!

Initially, the system is at rest and there is no kinetic energy ($KE_i = 0$), whereas the initial potential energy (per unit length in the transverse direction) is

$$PE_i = \frac{1}{2}\,\rho_0 \int_{-\infty}^{0} g'H^2\,dx. \tag{13-16}$$

(Although this expression is infinite, only the difference with the final potential energy will interest us. So, there is no problem.) At the final state, the velocity u is zero, leaving the kinetic energy to be

$$KE_f = \frac{1}{2} \rho_0 \int_{-\infty}^{R} h v^2 \, dx, \tag{13-17}$$

and the potential energy is

$$PE_f = \frac{1}{2} \rho_0 \int_{-\infty}^{R} g' h^2 \, dx. \tag{13-18}$$

During the spreading phase, some of the lighter water has been raised and some heavier water has been lowered to take its place. Hence, the center of gravity of the system has been lowered, and we expect a drop in potential energy. Calculations yield

$$\Delta PE = PE_i - PE_f = \tfrac{1}{4} g' H^2 R. \tag{13-19}$$

Some kinetic energy has been created by setting a transverse current; the amount is

$$\Delta KE = KE_f - KE_i = \tfrac{1}{12} g' H^2 R. \tag{13-20}$$

Therefore, as we can clearly see, only one-third of the potential-energy drop has been consumed by the production of kinetic energy, and the following question arises: What has happened to the other two-thirds of the released potential energy? The answer lies in the presence of transients, which occur during the adjustment: some of the time-dependent motions are gravity waves (here, internal waves on the interface), which travel to infinity, thus effectively carrying energy away from the region of adjustment. In reality, such waves dissipate along the way, and there is a net decrease of energy in the system. The ratio of kinetic-energy production to potential-energy release varies from case to case (Ou, 1986).

An interesting property of the geostrophically adjusted state is that it corresponds to the greatest energy loss and thus to a level of minimum energy. Let us demonstrate this proposition in the particular case at hand. The energy of the system is at all times

$$E = PE + KE = \frac{\rho_0}{2} \int_{-\infty}^{a} [g' h^2 + h (u^2 + v^2)] \, dx, \tag{13-21}$$

and we know that the evolution is constrained by conservation of potential vorticity:

$$f + \frac{\partial v}{\partial x} = \frac{f}{H} h. \tag{13-22}$$

Let us now search for the state that corresponds to the lowest possible level of energy, (13-21), under constraint (13-22) by forming the variational principle:

$$\mathcal{F}(h, u, v, \lambda) = \frac{\rho_0}{2} \int_{-\infty}^{0} \left[g' h^2 + h (u^2 + v^2) - 2\lambda \left(f + \frac{\partial v}{\partial x} - \frac{f}{H} h \right) \right] dx, \tag{13-23a}$$

$$\delta \mathcal{F} = 0 \text{ for any } \delta h, \ \delta u, \ \delta v, \text{ and } \delta\lambda. \tag{13-23b}$$

Because expression (13-21) is positive definite, the extremum will be a minimum. The variations with respect to the three state variables h, u, and v and the Lagrangian multiplier λ yield, respectively:

$$\delta h: \quad g'h + \frac{1}{2}(u^2 + v^2) + \frac{f}{H}\lambda = 0 \tag{13-24}$$

$$\delta u: \quad hu = 0 \tag{13-25}$$

$$\delta v: \quad hv + \frac{\partial \lambda}{\partial x} = 0 \tag{13-26}$$

$$\delta \lambda: \quad f + \frac{\partial v}{\partial x} - \frac{f}{H} h = 0. \tag{13-27}$$

Equation (13-25) provides $u = 0$, whereas the elimination of λ between (13-24) and (13-26) leads to

$$\frac{\partial}{\partial x}\left(g'h + \frac{1}{2} v^2\right) + \frac{f}{H}(-hv) = 0,$$

or

$$g' \frac{\partial h}{\partial x} + v\left(\frac{\partial v}{\partial x} - \frac{f}{H} h\right) = 0.$$

Finally, use of (13-27) reduces this last equation to

$$g' \frac{\partial h}{\partial x} - fv = 0.$$

In conclusion, the state of minimum energy is the state in which u vanishes, and the cross-isobaric velocity is geostrophic—that is, the steady, geostrophic state.

It can be shown that the preceding conclusion remains valid in the general case of arbitrary, multilayer potential-vorticity distributions, as long as the system is uniform in one horizontal direction. Therefore, it is a general rule that geostrophically adjusted states correspond to levels of minimum energy. This may explain why geophysical flows commonly adopt a nearly geostrophic balance.

PROBLEMS

13-1. In a certain region, at a certain time, the atmospheric temperature along the ground decreases northward at the rate of 1° C every 35 km, and there are good reasons to assume that this gradient does not change much with height. If there is no wind at ground level, what are the wind speed and direction at an altitude of 2 km? To answer, take latitude = 40°N, mean temperature = 290 K, and uniform pressure on the ground.

13-2. A cruise to the Gulf Stream at 38°N provided a cross-section of the current, which was then approximated to a two-layer model (Figure 13-5) with a warm layer of density

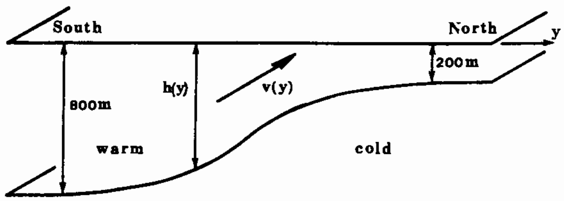


Figure 13-5 Schematic cross section of the Gulf Stream (Problem 13-2).

$\rho_1 = 1025$ kg/m^3 and depth $h(y) = H - \Delta H \tanh(y/L)$, overlying a colder layer of density $\rho_2 = 1029$ kg/m^3. Taking $H = 500$ m, $\Delta H = 300$ m, and $L = 60$ km and assuming that there is no flow in the lower layer and that the upper layer is in geostrophic balance, determine the flow pattern at the surface. What is the maximum velocity of the Gulf Stream? Where does it occur? Also, compare the jet width (L) to the radius of deformation.

13-3. Derive the discrete Margules relation (13-5) from the governing equations written in the density-coordinate system (Chapter 12).

13-4. Through the Strait of Gibraltar, connecting the Mediterranean Sea to the North Atlantic Ocean, there is an inflow of Atlantic waters near the top and an equal outflow of much more saline Mediterranean waters below. At its narrowest point (Tarifa Narrows), the strait is 11 km wide and 650 m deep. The stratification closely resembles a two-layer configuration with a relative density difference of 0.2% and an interface sloping from 175 m along the Spanish coast (north) to 225 m along the African coast (south). Taking $f = 8.5 \times 10^{-5}$ s^{-1}, approximating the cross-section to a rectangle, and assuming that the volumetric transport in one layer is equal and opposite to that in the other layer, estimate this volumetric transport.

13-5. Determine the geostrophically adjusted state of a band of warm water as depicted in Figure 13-4c. The variables are ρ_0 = density of water below, $\rho_0 - \Delta\rho$ = density of warm water, H = initial depth of warm water, $2a$ = initial width of warm-water band, and $2b$ = width of warm-water band after adjustment. In particular, determine the value b, and investigate the limits when the initial half-width a is much less and much greater than the deformation radius R.

13-6. Find the solution for the geostrophically adjusted state of the initial configuration shown on Figure 13-6, and calculate the fraction of potential-energy release that has been converted into kinetic energy of the final steady state.

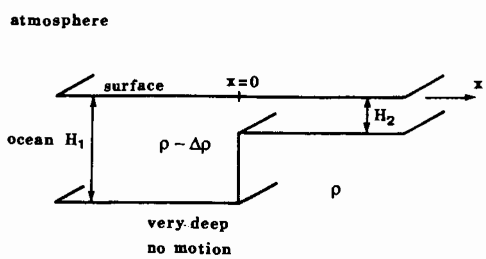


Figure 13-6 State prior to geostrophic adjustment (Problem 13-6).

13-7. In a valley of the French Alps ($\simeq 45°N$), one village (A) is situated on a flank 500 m above the valley floor and another (B) lies on the opposite side 200 m above the valley floor (Figure 13-7). The horizontal distance between the two is 40 km. One day, a shepherd in the upper village, who is also a fine meteorologist, notes a cold wind with temperature 6° C. Upon calling her cousin, a blacksmith in the lower village across the

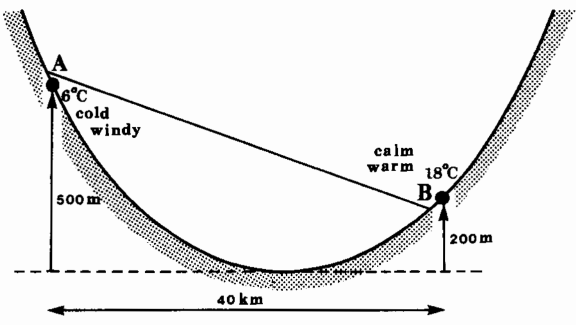


Figure 13-7 Air masses in a valley: What is the wind? (Problem 13-7).

valley, she learns that he is enjoying a calm afternoon with a comfortable 18° C! Assuming that the explanation of this perplexing situation resides in a cold wind blowing along one side of the valley (Figure 13-7), she is able to determine a lower bound for its speed. Can you? Also, in which direction is the wind blowing? (*Hint*: Do not ignore compressibility of air.)

SUGGESTED LABORATORY DEMONSTRATION

Equipment

Rotating table with cylindrical container, cylindrical tube substantially narrower than the container, two fluids of different densities and color (e.g., dyed salt water and fresh water).

Experiment

Place the tube vertically in middle of the container. Fill it halfway with the heavier, dyed fluid. Fill the rest of the tube and container with the lighter fluid to same level. Bring the setup to rotation and wait until solid-body rotation is reached. Carefully remove tube vertically, releasing the heavier fluid. Watch that fluid spread along the bottom over a limited distance. Sprinkle flakes on the surface to detect a vortex motion.

George Veronis

1926 –

An applied mathematician turned oceanographer, George Veronis has been a driving force in geophysical fluid dynamics since its early days. With Willem V. R. Malkus, he cofounded the GFD Summer Program at the Woods Hole Oceanograhic Institution, which continues after more than thirty years to bring together oceanographers, meteorologists, physicists, and mathematicians to debate problems related to geophysical flows. Veronis is best known for his theoretical studies on oceanic circulation, rotating and stratified fluids, thermal convection with and without rotation, and double diffusive processes. His model of the circulation of the world ocean was an analytical study based on planetary geostrophic dynamics and the nonlinearity of thermal processes, in which he showed how western boundary currents cross the boundaries of wind gyres and connect all of the world's oceans into a single circulating system. Veronis has earned a reputation as a superb lecturer, who can explain difficult concepts with amazing ease and clarity. (*Photo credit: G. Veronis.*)