

Feb. - 28, 2012

Chapters 3 + 4

Bushman - Louis (2010)

# Governing Equations

p. - 16

Eulerian / Lagrangian perspectives, velocity field

p. - 18

Scaling momentum  $\rightarrow$  pressure dominates all  
introduce perturbation pressure  
and density

p.

p. 22/23 Boussinesq Mass balance

p. 24/25 apply Boussinesq to momentum (a) horizontal

(b) vertical

class #5



p. 26 a-c

Scaling

p. 27

Summary

p. 28

Energy equation temperature

p. 29

Equation of state  $\rho = \rho(T, S, p)$

skipped formal derivation of continuity (p. 21)

skipped formal derivation of stress tensor

Newtonian fluid stress  $\propto$  strain

} Appendix A

in book

class Fluid Dynamics

p. 30 formal scaling, Non-dimensional numbers

p. 31

Rossby number of deformation

## Governing Equations

Newton's 2<sup>nd</sup> law in a rotating frame

$$\frac{d\vec{u}}{dt} + 2\vec{\Omega} \times \vec{u} = \sum_i \vec{F}_i$$

acceleration of fluid  
particle

sum of all forces  
applied to that particle

A fluid parcel is defined by

$$\vec{x} = \vec{x}(t, \vec{x}(t=0))$$

and has a velocity

$$\vec{u} = \vec{u}(t, \vec{x}(t=0))$$

or, if we tag it by its

$$\text{trajectory } \vec{x}_p(t)$$

$$\vec{u} = \vec{u}(t, x_p(t), y_p(t), z_p(t))$$

then the time-rate-of-change becomes

$$\vec{i} \cdot \frac{d\vec{u}}{dt} = \frac{d}{dt} u = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}$$

$$= \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z}$$

$$= \vec{i} \cdot \underbrace{\left( \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} \right)}_{\text{advection}}$$

$$\text{where } \vec{\sigma} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

Lagrangian

Eulerian  $\left\{ \begin{array}{ll} \frac{D}{Dt} \vec{u} & \text{advection} \\ & \text{(nonlinear)} \end{array} \right.$

We have interpret velocity as a velocity field

not used

Other, more formal derivations exist. The important part is to be able to distinguish between a statement that applies to discrete particles and their trajectories and statements that apply to velocity fields.

That said, at a given location and time, there is only one velocity vector, that is, both Eulerian and Lagrangian velocities are the same:

$$\vec{u}_p(t, \vec{x}(t=0), \vec{x}(t)) = \vec{u}(t, \vec{x})$$

$$\vec{u}_p = \frac{\partial \vec{x}_p}{\partial t} = \vec{u}(\vec{x}, t) = \vec{u}(\vec{x}_p, t)$$

this is a differential equation, i.e.,

the Eulerian velocity field  $\vec{u}$  must be known

to find the location of a particle as a function of time

$$\vec{x}_p(t) - \vec{x}_p(t=0) = \int_{t=0}^t \vec{u}(\vec{x}, t) dt$$

Example: Stommed paper in Proc. Nat. Acad. Science (with Regret, I think)

Thus

$$\frac{D}{Dt} \vec{u} + 2\vec{\Omega} \times \vec{u} = -\frac{1}{\rho} \vec{\nabla} p + \vec{f}_{\text{stress}} - \vec{\nabla} (\Xi_g - \frac{1}{2} \vec{\Omega}^2 \vec{k} \cdot \vec{k}/2) \quad \Xi_c$$

accelerations

pressure gradient  
friction "normal stress"  
gravity "tangential stress"  
centrifugal potentials  
appendix - A

$$\frac{U}{T^2}$$

$$\frac{24}{\pi}$$

$$\frac{P}{\rho}$$

?

$$g$$

Other: 10 m/s

(24.3600)s

$$10 \text{ m/s}$$

24.3600 seconds

$$10 \text{ m/s}^2$$

1 second

Ocean: 1 m/s

day

$$1 \text{ m/s}$$

day

$$10 \text{ m/s}$$

1 sec

Thus in geophysical flows gravity is usually several orders of magnitudes larger than the accelerations

Hence we define perturbation pressure  $\overset{\circ}{p}$  and perturbation density  $\overset{\circ}{\rho}$  as departures from an equilibrium (at rest  $\overset{\circ}{p} = 0, \overset{\circ}{\rho} = 0$ ) :

$$p = p_0(z) + p'$$

$$\rho = \rho_0(z) + \rho'$$

that satisfy

$$\frac{dp_0}{dz} = -g \rho_0$$

in the vertical

(hydrostatic)

(will return)  
to this later

The perturbation pressure and density then are

$$p = p_0(z) \pm p'(x, y, z, t)$$

$$\rho = \rho_0(z) \pm \rho'(x, y, z, t)$$

actual @ rest      perturbation due to motion

$$p_0 \gg p'$$

$\downarrow$   
not used

and our momentum balance becomes

after  
Boussinesq

$$\frac{D}{Dt} \vec{u} + 2 \vec{\Omega} \times \vec{u} = -\frac{1}{\rho_0} \vec{\nabla} p' - \frac{\rho'}{\rho_0} \vec{g} + \vec{F}_{\text{stress}}$$

This is the  
Boussinesq  
Approximation

In the special case of a homogeneous fluid (uniform density)

$$p' = 0$$

(barotropic dynamics  
no buoyancy forces)

Otherwise  $-\rho' \vec{g}$  represents the buoyancy force (baroclinic dynamics)

Independent variables :  $t, x, y, z$

Dependent variables :  $u, v, w, p, \rho$

3 equations for 5 dependent variables

→ need two more independent equations to determine the system's properties

For a fluid parcel Newton's 2<sup>nd</sup> law in rotating frame states that

$$\frac{d\vec{u}}{dt} + 2\vec{\omega} \times \vec{u} = \sum_i \vec{F}_i$$

define a fluid parcel by its trajectory, that is,

$$\vec{x}_p = \vec{x}_{p_0}(t, \vec{x}(t-t_0))$$

it has a velocity

$$u_p = \frac{d\vec{x}_p}{dt} = \frac{d\vec{x}_p}{dt}(t, \vec{x}(t-t_0)) = \vec{u}(t, \vec{x}_p(t-t_0)) = \vec{u}(t, \vec{x}_p(t))$$

$\uparrow$  differential equation for  $\vec{x}_p$        $\uparrow$        $\downarrow$

note that there

is only one velocity, i.e.g.,  
~~fixed~~

$$\vec{u}_p(t, \vec{x}_p(t)) = \vec{u}(t, \vec{x})$$

velocity of a  
parcel that at  
time  $t$  occupies locati  
 $\vec{x}$

velocity at time  $t$   
at location  $\vec{x}$

fixed parcel  
(changing location)  
Lagrangian

fixed location  
(changing with time)  
Eulerian velocity

$$\vec{x}_p(t) = \vec{x}_p(t_0) + \int_{t_0}^t \vec{u}(\vec{x}, t) dt$$

16

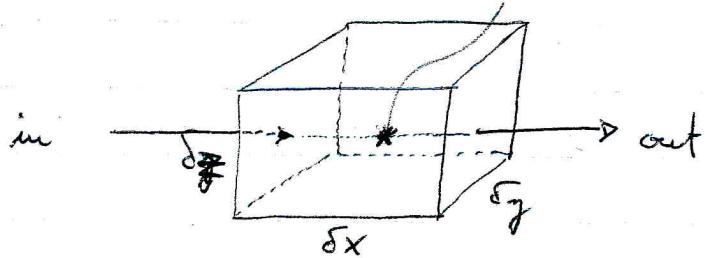
$$\frac{d\vec{u}_p}{dt} = \frac{d\vec{u}_p}{dt} + (\vec{u}_p \cdot \vec{\omega}) \vec{u}_p \equiv \frac{D}{Dt} \vec{u}$$

then conservation of mass ②

then Bernoulli's law ③  
then  $m = m_0$

## Conservation of Mass

Consider a volume fixed in space center point



$$\text{in flow: } \frac{\delta m}{\delta y \delta z} = \left[ \rho u - \frac{1}{2} \frac{\partial (\rho u)}{\partial x} \delta x \right] \delta y \delta z$$

$$\text{out : } \frac{\delta m}{\delta y \delta z} = \left[ \rho u + \frac{1}{2} \frac{\partial (\rho u)}{\partial x} \delta x \right] \delta y \delta z$$

$$\text{in-out through } \delta y \delta z = - \frac{\partial (\rho u)}{\partial x} \delta x \delta y \delta z$$

in-out all sides

$$- \vec{\nabla}(\rho \vec{u}) \cdot \delta x \delta y \delta z$$

As  $\delta x \delta y \delta z = \delta V \rightarrow 0$  the rate of increase of mass per unit volume becomes

$$- \vec{\nabla}(\rho \vec{u}) = \frac{d\rho}{dt} \quad \text{because } \rho \text{ is mass unit/volume}$$

or, finally

$$\frac{d\rho}{dt} + \vec{\nabla}(\rho \vec{u}) = 0 \quad \text{mass conservation}$$

go to p. 22 to apply Bernoulli's law.

(no 4 equations for 5 variables)

## Boussinesq Approximation

density of freshwater  $\sim 1000 \text{ kg/m}^3$

$\rho_{\text{max}}$  in ocean  $\sim 1040 \text{ kg/m}^3$

(Red Sea)

Hence it appears reasonable to assume

$$\rho = \rho_0 + \rho' \quad \text{with } \rho' \ll \rho_0$$

$$40 \ll 1000$$

What about atmosphere where  $\rho \in [0, 1] \text{ kg/m}^3$ ?

$$\rho = \rho_0(z) + \rho'(x, y, z, t) \quad \text{with } \frac{\partial \rho_0}{\partial z} = -\rho_0 g$$

$$\rho = \rho_0(z) + \rho'(x, y, z, t)$$

This will effectively lead to the same approximations by considering density and pressure perturbations from a stably stratified density distribution  $\rho_0(z)$  balanced by a hydrostatic pressure  $\rho_0$ .

Consider mass balance

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{u}) = 0 \quad \text{with } \rho = \rho_0 + \rho'$$

$$\cancel{\frac{\partial \rho_0}{\partial t}} + \cancel{\frac{\partial \rho'}{\partial t}} + (\rho_0 + \rho') \vec{\nabla} \cdot \vec{u} + (\vec{u} \cdot \vec{\nabla})(\rho_0 + \rho') = 0$$

$$\frac{1}{\rho} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{u} + (\vec{u} \cdot \vec{\nabla}) \rho = \frac{1}{\rho} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{u} = 0 \quad \text{or} \quad \frac{\partial(\rho)}{\partial t} + \rho \vec{\nabla} \cdot \vec{u} = 0$$

(24)

$$\frac{\partial \rho}{\partial z} \text{ at most, but } \frac{\partial \rho_0}{\partial z} \sim \rho' \frac{H}{T} \frac{1}{H} \quad (23) \\ \sim \rho'/T$$

$$\text{because } \frac{\Delta P_0}{\Delta z} \sim \frac{\rho'}{H}$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{u} + \rho' \vec{\nabla} \cdot \vec{u} + (\vec{u} \cdot \vec{\nabla}) \rho_0 + (\vec{u} \cdot \vec{\nabla}) \rho' = 0$$

$$\rho_0 u/L - \rho' u/L$$

$$\rho' u/L$$

$$\frac{\rho'}{T}, \rho_0 \cancel{\perp \frac{L}{T}}, \rho' \cancel{\perp \frac{L}{T}}, \rho' \cancel{\frac{H}{T}} \cancel{\perp \frac{L}{T}} \rho'$$

$$\frac{\rho'}{\rho_0} \perp \frac{\rho'}{\rho_0} \perp \frac{\rho'}{\rho_0} \perp \frac{\rho'}{\rho_0}$$

$$\therefore \rho_0 \vec{\nabla} \cdot \vec{u} = 0 \quad \text{or} \quad \vec{\nabla} \cdot \vec{u} = 0$$

Hence in components

$$\underbrace{\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + f \cos \phi - f \sin \phi \right)}_{(\rho_0 + \rho')}(v) = - \frac{\partial p}{\partial x} + \text{friction}$$

$$\text{but } \rho = \rho_0(z) + \rho'(x, y, z, t) \quad \text{with } \rho_0 \gg \rho'$$

$$\text{and } \rho = \rho_0(z) + \rho_0^*(x, y, z, t) \quad \text{with } \frac{\partial \rho_0}{\partial z} = -\rho_0 g$$

this gives

$$\rho_0^* \left( \frac{\partial u}{\partial t} + u \cdot (\vec{u} \cdot \vec{\nabla}) \vec{u} + w \cos \phi - f v \sin \phi \right) = - \frac{\partial \rho_0^*}{\partial x} + \text{friction}$$

perturbation pressure as  $\frac{\partial \rho_0}{\partial x} = 0$   
 $\rho_0 = \rho_0(z)$

other horizontal component is similar, but the vertical component is more tricky

$$\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} - f \cos \phi u \right) = - \frac{\partial p}{\partial z} - \rho g + \text{friction}$$

$$(\rho_0 + \rho') \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} - f \cos \phi u \right) = - \frac{\partial \rho_0}{\partial z} - \frac{\partial \rho'}{\partial z} - \rho_0 g - \rho' g + \text{friction}$$

$\boxed{= 0 \text{ by definition}}$

we cannot neglect  
this term

→ buoyancy force

$$\frac{\rho'}{\rho_0} g$$

Hence our momentum balance becomes

$$\frac{D \vec{u}}{Dt} + 2\vec{\Omega} \times \vec{u} = -\frac{1}{\rho_0} \vec{\nabla} p' - \frac{\rho'}{\rho_0} \vec{g} + \vec{f}_{\text{friction}}$$

In the special case of  $\vec{f} = 0$  (homogeneous fluid, uniform density)

$$p' = 0$$

→ barotropic dynamics, no buoyancy forces

Otherwise  $-\frac{\rho'}{\rho_0} \vec{g}$  represents the buoyancy force  
(reduced gravity)

→ baroclinic dynamics

Next: Scale these equations using  $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial w}{\partial z}$

that is  $\frac{u}{L} \sim \frac{w}{H}$

or  $w \sim u \frac{H}{L}$

or  $w \ll u$  as  $H \ll L$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + 2\Omega \cos\phi \cdot w - 2\Omega \sin\phi \cdot u = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \text{friction}$$

(28) (26)

perturbations  
pressure

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$W \sim \frac{H \cdot U}{L}$$

$\frac{u}{T}$	$\frac{u^2}{L}$	$\frac{u^2}{L}$	$\frac{W^2 \cdot U}{H}$	$2\Omega W$	$2\Omega \cdot U$	$\frac{P}{\rho_0 L}$	$\frac{vU}{L^2}$	$\frac{vW}{H^2}$
$\frac{1}{\Omega T}$		$\frac{u^2}{\Omega L}$		$2\Omega \frac{W}{U}$	$2\Omega$	$\frac{P}{\rho_0 L \cdot U \cdot 2\Omega}$	$\frac{vW \cdot M}{L^2 \cdot H^2}$	

$R_{\theta_T}$	$R_\theta$	<del>1</del>	1	?	$\frac{vU}{H^2 \cdot 2\Omega}$
					$[v] = \frac{m^2}{s}$ $E_v$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} - 2\Omega \cos\phi u = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} - \frac{g p}{\rho_0} + \text{friction}$$

$$[v] = \frac{m^2}{s} \quad E_v$$

$\frac{w}{T}$	$\frac{uw}{L}$	$\frac{uw}{L}$	$\frac{w^2}{H}$	$2\Omega U$	$\frac{P}{\rho_0 H}$	$\frac{g \Delta p}{\rho_0}$	$\frac{vW}{L^2} \ll \frac{vW}{H^2}$
$\frac{1}{T}$		$\frac{w^2}{H}$		$2\Omega \frac{u \cdot L \cdot w}{K \cdot H}$	$\frac{P}{\rho H \cdot W}$	$g \frac{\Delta p}{\rho} W$	$\frac{v}{H^2} \ll \frac{vU}{H^2} \sim \frac{vU}{\rho \cdot u}$

<del>1</del>	<del>1</del>	$2\Omega \frac{L}{H}$	$\frac{P}{\rho H^2 \cdot \Omega}$	$g \frac{\Delta p}{\rho} \frac{T}{H \cdot \Omega}$	$\frac{v}{H^2}$
$\geq \Omega$	$\geq \Omega$	$\gg \Omega$	?	?	?

$$\begin{aligned} \Omega &\sim 10^{-5} \text{ s}^{-1} \\ U &\sim 10 \text{ m s}^{-1} \\ g &\sim 10 \text{ m s}^{-2} \\ \Delta p &\sim 10 \text{ kg m}^{-3} \\ \rho &\sim 10^3 \text{ kg m}^{-3} \end{aligned} \left\{ \begin{aligned} \frac{2\Omega U}{g \Delta p / \rho} &\sim \frac{10^{-5}}{10^2 \cdot 10^3} \sim 10^{-4} \end{aligned} \right.$$

$$1 \Rightarrow \frac{2\Omega U}{\rho \Delta p H} \quad \frac{P}{\rho H^2 \cdot \Omega} \quad 1 \quad \frac{v}{\sqrt{\Delta p H}}$$