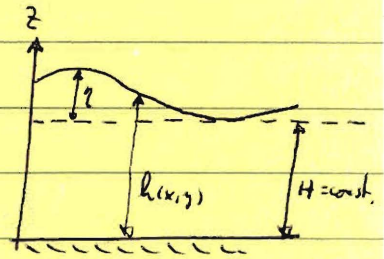
Show movie on Rossby Adjustment (U. Rhode Island)

Recall non-dimensional shallow water equation, $E_v \ll 1$, $\rho' = \sigma$

$$Ro_T\left(\frac{\partial u}{\partial t}\right) + Ro\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - v = -g\frac{\partial \eta}{\partial x}$$



Assume $Ro_T \sim 1$ $\Rightarrow$ $f \sim 1/T$

and $Ro = \frac{U}{fL} \sim \frac{U}{L/T} \doteq \frac{\text{particle velocity}}{\text{phase}} = \frac{U}{C} \ll 1$

↳ Froude number

Here we consider "linear" waves, that is $Ro = Froude \ll 1$

and the (dimensional) momentum balance becomes

$$\underbrace{\frac{\partial u^*}{\partial t^*}}_{\sigma(Ro_T)} - \underbrace{f v^*}_{\sigma(1)} = \underbrace{-g\frac{\partial \eta^*}{\partial x^*}}_{\sigma(1)} \qquad \frac{\partial v^*}{\partial t^*} + f u^* = -g\frac{\partial \eta^*}{\partial y^*}$$

linearized momentum

The continuity equation

$$\frac{\partial h^*}{\partial t^*} + \frac{\partial (u^* h^*)}{\partial x^*} + \frac{\partial (v^* h^*)}{\partial y^*} = \sigma$$

becomes

$h = H + \eta$

$\eta^* = \Delta H \cdot \eta$

$u^* = U u$

$$\underbrace{\frac{\partial \eta^*}{\partial t^*}}_{\text{linear}} + \underbrace{\left(u^*\frac{\partial \eta^*}{\partial x^*} + v^*\frac{\partial \eta^*}{\partial y^*}\right)}_{\text{nonlinear}} + \underbrace{H\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*}\right)}_{\text{linear}} + \underbrace{\eta^*\left(\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*}\right)}_{\text{nonlinear}} = \sigma$$

in non-dimensional form

$$\underbrace{\frac{\Delta H}{H}\left(\frac{\partial \eta}{\partial t}\right)}_{O\left(\frac{\Delta H}{H}\right)} + \underbrace{\frac{\Delta H}{H} Ro\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right)}_{O\left(Ro\,\frac{\Delta H}{H}\right)} + \underbrace{Ro\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{O(Ro)} + \underbrace{\frac{\Delta H}{H} Ro\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{O\left(Ro\,\frac{\Delta H}{H}\right)} = 0$$

$\frac{\Delta H}{H} \sim Ro \ll 1$

and for $\frac{\Delta H}{H} \sim Ro \ll 1$

we get

$$\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

in dimensional form

$$\boxed{\frac{1}{H}\frac{\partial \eta^*}{\partial t^*} + \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0}$$

"linearized" continuity equation

together with momentum we have 3 PDE for 3 variables (u, v, η)

Lets drop the * and consider all variables dimensionally,

(A) take $\frac{\partial}{\partial x}\left(x\text{-momentum}\right) + \frac{\partial}{\partial y}\left(y\text{-momentum}\right)$

use $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{1}{H}\frac{\partial \eta}{\partial t}$ from continuity

vorticity

$$\frac{\partial}{\partial t}\left(\frac{\xi}{H} - \frac{\eta}{H}\frac{f}{H}\right) = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 \eta}{\partial t^2} - c^2\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) + fH\xi = 0}$$

2 equation for 2 unknown (η, ξ)

where $c^2 \equiv gH$ is the phase velocity, and

$\xi \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the relative vorticity (local spin of fluid) (f is planetary vorticity) ambient spin of earth

For $f = 0$ this is a "classical" wave equation ~~for~~ giving sinusoidal solutions (later)

but for $f \neq 0$ (presence of rotation) we must consider relative vorticity, so we "take the curl":

Gill (1982) chapter-7

(B) $\frac{\partial}{\partial y}$ (x-momentum) $-$ $\frac{\partial}{\partial x}$ (y-momentum)

$\frac{1}{H^2}\frac{\partial \xi}{\partial t} = \frac{f}{H^2}\frac{\partial \eta}{\partial t}$

$\frac{\partial}{\partial t}\left(\frac{\xi}{H} - \frac{f}{H^2}\eta\right) = 0$

and we get

$$\frac{1}{f}\frac{\partial \xi}{\partial t} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)$$

$$= +\frac{1}{H}\frac{\partial \eta}{\partial t}$$ using continuity

add the missing step

$\frac{\partial}{\partial t}(H \cdot \xi) - \frac{\partial}{\partial t}(f\eta) = 0$

$$\frac{\partial}{\partial t}\left[\frac{\xi + \eta/H f}{H}\right] = 0$$

$$\frac{\partial}{\partial t}\left(\frac{\xi}{H} + \frac{\eta f}{H^2}\right) = 0$$

relative vorticity — vortex tube stretching

this is the "linearized form of the fully nonlinear potential vorticity $\frac{D}{Dt}\left(\frac{f+\xi}{h}\right) = 0$

vorticity has units of $(ms)^{-1}$

integrate once with respect to time

$$Q \equiv \frac{\xi}{H} - \frac{\eta f}{H^2} = \frac{\xi}{H} - \frac{\eta}{H}\left(\frac{f}{H}\right)$$

this gives

$$Q(x,y,t) = Q(x,y,t=0)$$

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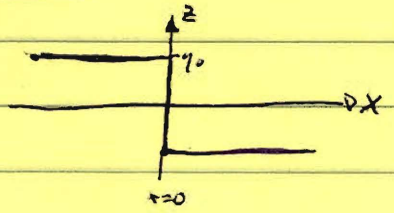
Example: for $u = v = 0$ and $\eta = -\eta_0\, \mathrm{sgn}(x)$ at $t = 0$



$$\frac{\xi}{H} - \frac{\eta f}{H^2} = \frac{\eta_0}{H^2}\,\mathrm{sgn}(x)$$

so $$\xi = f\left(\frac{\eta_0}{H}\,\mathrm{sgn}(x) + \frac{\eta}{H}\right)$$

$Q = \frac{\xi H - \eta f}{H^2}$

put this expression of relative vorticity into the forced wave equation (A):

$$\frac{\partial^2 \eta}{\partial t^2} - c^2\left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2}\right) + f H\left(f\left(\frac{\eta_0}{H}\,\mathrm{sgn}(x) + \frac{\eta}{H}\right)\right) = 0$$

or

$$\left[\frac{\partial^2}{\partial t^2} - c^2\left(\frac{\partial^2 \cdot}{\partial x^2} + \frac{\partial^2 \cdot}{\partial y^2}\right)\right]\eta + f^2\eta = -f^2 \eta_0\,\mathrm{sgn}(x)$$

$$\left[\frac{\partial^2 \cdot}{\partial t^2} - c^2\frac{\partial^2 \cdot}{\partial x^2} + \frac{\partial^2 \cdot}{\partial y^2} + f^2 \cdot\right]\eta = -f^2 \eta_0\,\mathrm{sgn}(x)$$

we have one partial differential equation for one variable ! → $\eta = \eta(x,t)$ $\xi = f\left(\frac{\eta_0}{H}\,\mathrm{sgn}(x) + \frac{\eta}{H}\right)$

That's a substantial reduction from 3 PDE for 3 variables

In a "final" steady state $\frac{\partial \cdot}{\partial t} = 0$ and the above vorticity equation becomes

$$-c^2 \nabla^2 \eta + f^2 \eta = -f^2 \eta_0\,\mathrm{sgn}(x)$$

or $$-c^2 \frac{\partial^2 \eta}{\partial x^2} + f^2 \eta = -f^2 \eta_0\,\mathrm{sgn}(x)$$ if $\frac{\partial \cdot}{\partial y} = 0$ since the initial condition was one in "x" only

$$\eta'' - \left(\frac{f}{c}\right)^2 \eta = \underbrace{f^2 \eta_0\,\mathrm{sgn}(x)}_{\text{"forcing"}}$$

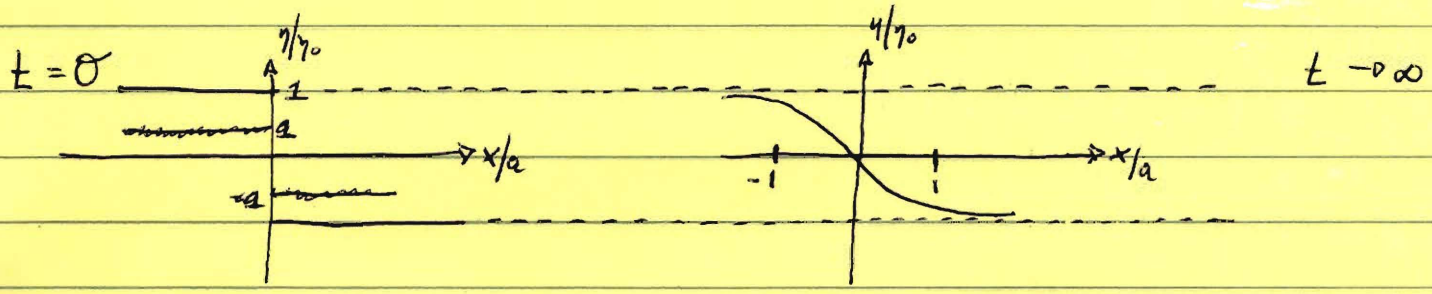
2nd order ODE

which has solutions

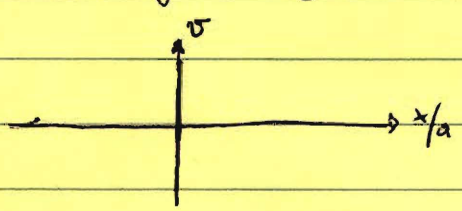
$$\eta(x) = \eta_0 \begin{cases} -1 + e^{-x/a} & \text{for } x > 0 \\ +1 - e^{+x/a} & \text{for } x < 0 \end{cases}$$

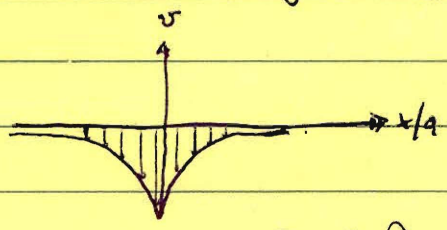
where $a = \dfrac{c}{|f|} = \sqrt{gH}\Big/|f|$

is the Rossby radius of deformation



steady state @ $t=0$ steady state @ $t \to \infty$





$$v = \frac{g}{f}\frac{\partial \eta}{\partial x} = \frac{-g\eta_0}{fa} e^{-|x|/a}$$

The conservation of potential vorticity allows us to calculate a new (geostrophic) steady state from an initial state of rest

Note that it was NOT possible to calculate this new steady state from the steady state equations, as the geostrophic balance

$$-fv = -g\frac{\partial \eta}{\partial x} \quad \text{and} \quad +fu = -g\frac{\partial \eta}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

with stream functions $u \equiv \dfrac{\partial \psi}{\partial y}$ and $v \equiv -\dfrac{\partial \psi}{\partial x}$ give

$$f\frac{\partial \psi}{\partial x} = -g\frac{\partial \eta}{\partial x} \qquad f\frac{\partial \psi}{\partial y} = -g\frac{\partial \eta}{\partial y} \qquad \text{and} \qquad \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} = 0 \ \checkmark$$

stream function satisfies continuity

~~$f\,\psi(x,y) = -g\,\eta(x,y)$~~

that suggest

$$\frac{\partial}{\partial y}(x\text{-mom}) - \frac{\partial}{\partial x}(y\text{-mom})$$

$$f\frac{\partial^2 \psi}{\partial x \partial y} + g\frac{\partial^2 \eta}{\partial x \partial y} = f\frac{\partial^2 \psi}{\partial y \partial x} + g\frac{\partial^2 \eta}{\partial y \partial x} = 0$$

$$f\left(\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right) + g\left(\frac{\partial^2 \eta}{\partial x^2} - \frac{\partial^2 \eta}{\partial y^2}\right)$$

$$0 = 0$$

~~degenerate~~, the two geostrophic equations always satisfy the continuity equation, that is, continuity is not an independent linear equation. Any distribution of η gives a stream function, e.g.,

$$\frac{\partial x\text{-mom}}{\partial x} + \frac{\partial y\text{-mom}}{\partial y} = \text{continuity}$$

linear independent ✓

Here we have only 2 equations for the three variables (u, v, η)

$$\psi = \frac{-g}{f}\eta$$