

G7.3

The complete problem is solved if we find complete solutions to

$$\frac{\partial^2 \eta}{\partial t^2} - c^2 \left( \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) + f^2 \eta = -f^2 \eta_0 \operatorname{sgn}(x)$$

relative velocity      stretching

Consider this complete solution to consist of two parts, that is

$$\eta(x, y, t) = \eta_H(x, y, t) + \eta_{\text{steady}}$$

general solution of

the forced (inhomogeneous)  
problem

general solution of

the unforced (homogeneous)  
problem

particular solution

of the forced  
problem

which gives

(a) forced problem

$$-\frac{\partial^2 \eta_{\text{steady}}}{\partial x^2} - \frac{\partial^2 \eta_{\text{steady}}}{\partial y^2} + f^2 \eta_{\text{steady}} = -f^2 \eta_0 \operatorname{sgn}(x)$$

(b) unforced problem

$$\frac{\partial^2 \eta_H}{\partial t^2} - c^2 \left( \frac{\partial^2 \eta_H}{\partial x^2} + \frac{\partial^2 \eta_H}{\partial y^2} \right) + f^2 \eta_H = 0$$

We solved (a) that gave us  $\eta_{\text{steady}} = \eta_0 \operatorname{sgn}(x) (-1 + e^{-|x|/q})$

We still need to solve (b) that give us the time dependent  
or transient part of the complete solution

For initial condition we also have

$$\eta(x, y, t=0) = \eta_H(x, y, t=0) + \eta_{\text{steady}}(x, y)$$

or

$$\eta_H(x, y, t=0) = -y_0 \operatorname{sgn}(x) - y_0 \operatorname{sgn}(x) (-1 + e^{-|x|/a})$$

$$\eta_H(x, y, t=0) = -y_0 \operatorname{sgn}(x) e^{-|x|/a}$$

The transients are obtained subject to this condition from

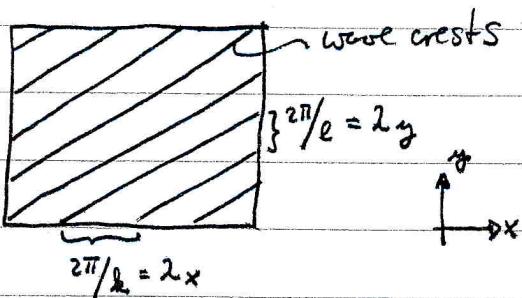
$$\frac{\partial^2 \eta_H}{\partial t^2} - c^2 \left( \frac{\partial^2 \eta_H}{\partial x^2} + \frac{\partial^2 \eta_H}{\partial y^2} \right) + f^2 \eta_H = 0$$

and solutions are of the form

$$\eta_H \propto \exp[i(kx + ly - \omega t)]$$

where  $\vec{k} = (k, l)$  is the horizontal vector wave number

$$\text{with } k_H^2 = k^2 + l^2 = |\vec{k}|$$



G8.2

Poincaré solutions

$$\frac{\partial u}{\partial t} - fv = -g \frac{\partial \gamma}{\partial x} \rightarrow -i\omega u - fv = -g i k \gamma$$

$$\frac{\partial v}{\partial t} + fu = -g \frac{\partial \gamma}{\partial y} \rightarrow -i\omega v + fu = -g i k \gamma$$

$$\frac{\partial \gamma}{\partial t} + H \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \rightarrow -i\omega \gamma + iHku + iHlv = 0$$

Try solutions  $\begin{pmatrix} u \\ v \\ \gamma \end{pmatrix} \propto \exp[i(kx+ly-\omega t)]$

3 algebraic equations  
for  $(u, v, \gamma)$

$$\therefore u = \frac{k\omega + ifl}{k_H^2 H} \gamma \rightarrow \frac{k\omega}{k^2 H} \gamma = \frac{\omega}{kH} \gamma$$

$$v = \frac{l\omega - ifk}{k_H^2 H} \gamma \rightarrow \frac{-ifk}{k^2 H} \gamma = \frac{-if}{kH} \gamma$$

orient co-ordinate system in the direction of progressive wave (i.e.,  $k=0$ )  
and  $\gamma = \gamma_0 \cos(kx - \omega t)$

gives

$$u = \frac{\omega \gamma_0 \cos(kx - \omega t)}{kH} \quad v = \frac{f \gamma_0 \sin(kx - \omega t)}{kH}$$

$$\text{thus } \frac{u^2 + v^2}{f^2(\gamma_0^2/kH)^2} = \frac{\omega^2 \cos^2 + f^2 \sin^2}{f^2} = \left(\frac{\omega}{f}\right)^2 \cos^2 + \sin^2$$

horizontal current ellipse with major/minor axis ratio  $\frac{\omega}{f}$ !

$$\omega^2 = f^2 + \kappa_H^2 c^2$$

(535)  
3-28-00

short Poincaré waves ( $\kappa_H a \gg 1$ )

have

$$\omega \approx \kappa_H c = k c$$

thus ellipticity  $\frac{\omega}{f} \approx \frac{k c}{f} = k \cdot a \gg 1$  long + thin  
almost rectangular

long Poincaré waves ( $\kappa_H a \ll 1$ )

have  $\omega \approx f$

thus ellipticity is  $\frac{\omega}{f} \approx \frac{f}{f} = 1$  circular motion

sense of rotation always clockwise (anti-cyclonic)

show Gill Fig. 8.2 and 8.3

then move to Fig. 9.3 : Forcing due to surface stress

That give the dispersion relation

note that the frequency  $\omega$   
is always larger than  
 $f$  (superinertial waves)

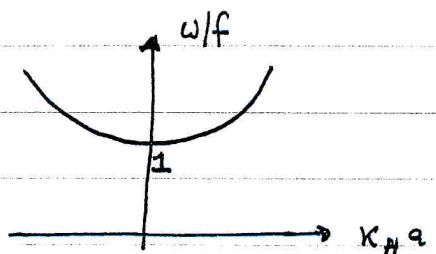
$$\omega^2 = f^2 + \kappa_H^2 c^2 \quad \text{where } \kappa_H^2 = k^2 + l^2$$

or

$$(\omega/f)^2 = 1 + (\kappa_H \cdot a)^2$$

$$c^2 = g H$$

$$a = c/f$$



Case a: Short waves  $\kappa_H a \gg 1$  waves are short rel.  
to Rossby radius

$$\omega \approx \kappa_H c$$

ordinary, non-dispersive, shallow water gravity waves  
that have phase speeds  $c_p$

$$c_p = \frac{\omega}{\kappa_H} = c = \sqrt{g H}$$

Case b: Long waves  $\kappa_H a \ll 1$  waves are long rel.  
to Rossby radius

$$\omega \sim f$$

wave with constant "inertial" frequency  
gravity not important.

→ Inertial oscillations

The group velocity of these waves is

$$\vec{c}_g = \left( \frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial \ell} \right) = \frac{c^2}{\omega} \vec{k}$$

for short waves

$$(x_H a \gg 1) \quad |c_g| \approx \frac{c^2 x_H}{\omega} \approx \frac{c^2 x_H}{x_H c} = c = \sqrt{g H'} \quad \underline{\text{FAST}}$$

for long waves

$$(x_H a \ll 1) \quad |c_g| \approx \frac{c^2 x_H}{f \omega} \approx \frac{c^2 x_H}{f} = c \cdot \underbrace{a x_H}_{\ll 1}$$

SLOW

which goes to 0 as  $x_H a \rightarrow 0$

That is, the short waves propagate away fast ( $\sqrt{g H'}$ ) while the longer waves stay behind, and in the limit do not propagate their energy at all!

$x_H a \rightarrow 0$

The transient, time dependent solution to the unforced solution consists of a superposition of such Poincaré waves. These will need to satisfy the initial conditions. They are symmetric about  $x=0$  and thus propagate in all directions. SHOW GILL SLIDES