

(83)

$$\frac{\partial \eta}{\partial t} - \alpha^2 \frac{\partial}{\partial t} \left( \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) - \beta_0 \alpha^2 \frac{\partial^2 \eta}{\partial x^2} = 0$$

solutions of the form  $\eta \propto \exp[i(lx + my - wt)]$   
gives

$$iw - \alpha^2 (iw) [(il^2)^2 + (im^2)^2] - \beta_0 \alpha^2 il = 0$$

$$w [1 - \alpha^2 (l^2 + m^2)] + \beta_0 \alpha^2 l = 0$$

$$\therefore w = \frac{-\beta_0 \alpha^2 l}{1 + \alpha^2 (l^2 + m^2)} = \frac{-\beta_0 l}{\alpha^{-2} + (l^2 + m^2)} \stackrel{m=0}{=} \frac{-\beta_0 l}{\alpha^{-2} + l^2}$$

$$c_x = \frac{\omega}{l} = \frac{-\beta_0 \alpha^2}{1 + \alpha^2 (l^2 + m^2)}$$

always negative  
that is westward phase  
propagation ALWAYS

fastest possible phase speed for  $(\alpha^2 c_{ph})^2 \ll 1$  or  $\alpha^2 \ll k_h^{-2}$  long waves  
is

$$c_{ph} = -\beta_0 \alpha^2 = -\frac{\beta_0}{f_0^2} g H \approx 10 \text{ m/s} \quad \text{in 1000 m of water}$$

$$\beta_0 = \frac{2 \Omega \cos \phi}{R} \approx \frac{7.292 \cdot 2 \cdot \cos \phi}{6300 \cdot 10^3 \text{ m s}} \approx \frac{f_0}{6 \times 10^6} \approx 10^{-11} \text{ m}^{-1} \text{s}^{-1}$$

$$\frac{\beta_0}{f_0^2} \approx \frac{10^{-4} \text{ m}^{-1} \text{s}^{-1}}{10^{-8} \text{ s}^{-2}} \approx 10^3 \text{ m}^{-1} \text{s} \quad c = \sqrt{g H} \approx 100 \text{ m/s} \quad \boxed{\frac{\beta_0 \cdot c^2}{f_0^2} \approx 10^{-3} \text{ m}^{-1} \text{s} \cdot \frac{10^4 \text{ m}^2 \text{s}^{-2}}{\text{s}^2}}$$

$$\approx 10 \frac{\text{m}}{\text{s}}$$

$$\omega = - \frac{\beta_0 R^2}{1 + R^2 k_H^2} k$$

$$(a k)^2 \gg 1 \quad \text{or} \quad a \gg k_H \quad \text{short waves} \quad \boxed{\omega \approx -\beta_0 \cdot k}$$

$$\rightarrow \frac{\omega}{f_0} \approx \frac{\beta_0 \cdot L}{f} = \beta \ll 1 \quad c_x = \frac{\omega}{k} \approx -\frac{\beta_0}{k^2} \quad \omega \approx -\beta_0 \left( \frac{k}{k^2 + l^2} \right)$$

$$(a k)^2 \gg 1 \quad \text{or} \quad a \ll k_H \quad \text{long waves} \quad \boxed{\omega \approx -\beta_0 R^2 k}$$

$$c_p = \frac{\omega}{k} = -\beta_0 R^2 \quad \text{non-dispersive}$$

$$c_x = \frac{\omega}{k} = -\beta_0 R^2$$

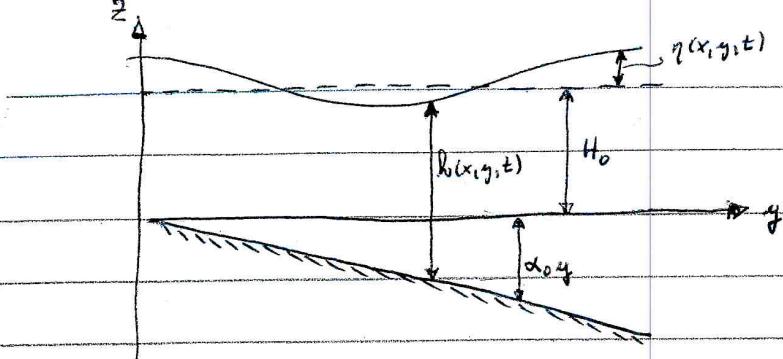
$$c_g = \frac{\partial \omega}{\partial k} = -\beta_0 R^2 = c_p$$

$$\frac{\omega}{f_0} \approx \frac{\beta_0 a^2}{f_0 L} = \left( \frac{\beta_0 \cdot L}{f_0} \right) \left( \frac{a}{L} \right)^2 \ll \beta \ll 1$$

$$\omega = - \frac{\beta_0 R^2 k}{1 + R^2 k^2 + l^2} = - \frac{\alpha}{\gamma}$$

$$= - \frac{\beta_0 k}{R^2 + k^2 + l^2} = - \frac{\beta k}{\alpha + k^2}$$

$$\alpha = \beta \frac{1}{R^2} + l^2$$



$$h = H_0 + \alpha_0 y + \eta(x, y, t)$$

with

$$\alpha = \frac{\alpha_0 L}{H} \ll 1$$

Continuity equation written (depth-integrated)

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} = 0$$

or

$$\frac{\partial \eta}{\partial t} + \left( u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} \right) + H_0 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \alpha_0 y \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \eta \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \alpha_0 v = 0$$

$$O\left(\frac{\Delta H}{H}\right)$$

$$O\left(R_o \frac{\Delta H}{H}\right)$$

$$O(R_o)$$

$$O(\alpha R_o)$$

$$O(R_o \frac{\Delta H}{H})$$

$$O(\alpha)$$

Hence for  $R_o \ll \frac{\Delta H}{H} \sim R_o \ll \alpha \ll 1$  ← Think of these conditions as "filters", e.g.,

$$R_{oT} = \frac{1}{\Omega T} \text{ or } \frac{af}{T}$$

we get

$$\frac{\partial \eta}{\partial t} + H_0 \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \alpha_0 v = 0$$

→  $\alpha \sim R_{oT}$  means that

$$\omega \sim \frac{1}{T} \sim \alpha \Omega \sim \frac{af}{T}$$

subinertial motions

Momentum equations and arguments the same as for the Rossby wave development, e.g.,

introduce  $u = u_0 + \alpha u_1 + O(\alpha^2)$  etc.

↓

$O(1)$  momentum is geostrophic

→  $O(\alpha)$  momentum is not geostrophic

$O(1)$  continuity is non-divergent

→  $O(\alpha)$  continuity is not divergent-free

These are the  
"linear quasi-geostrophic momentum  
equations"  
Gill (1982)  
p. 491

$$\sigma(d) \quad x\text{-momentum gives} \quad v_i = \dots \quad | \frac{\partial}{\partial y}$$

$$y\text{-momentum gives} \quad u_i = \dots \quad | \frac{\partial}{\partial x}$$

same as for Rossby wave

where the right hand side  
contains terms involving  
geostrophic velocity  $u_0, v_0$   
only

place these into the continuity to get

$$\frac{\partial \eta_i}{\partial t} - a^2 \frac{\partial}{\partial t} \nabla^2 \eta_0 - \beta_0 a^2 \frac{\partial \eta_0}{\partial x} + \alpha_0 a^2 \frac{\partial \eta_0}{\partial x} \cdot f = 0$$

Rossby Wave

\* new term due  
to sloping bottom

for the special case

that deep water is North  
and shallow water is  
South

For  $f = \text{const.}$  or  $\beta \ll \alpha \ll 1$

$$\frac{\partial \eta_i}{\partial t} - a^2 \frac{\partial}{\partial t} \nabla^2 \eta_0 + \alpha_0 a^2 \cdot f \frac{\partial \eta_0}{\partial x} = 0$$

$$\eta \propto \exp[i(kx + ly - \omega t)]$$

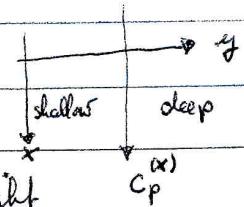
Dispersion relation is

$$\omega = \frac{\alpha_0 g}{f} \frac{k}{1 + a^2 (k^2 + l^2)}$$

is identical to that of the Rossby wave as  $\alpha_0$  carries a sign (as does  $f$ )  
indicating that the phase velocity in the  $x$ -direction

$$C_p^{(x)} = \frac{\omega}{k_x} = + \frac{\alpha_0 g}{f} \cdot \frac{1}{1 + a^2 k_N^2}$$

is positive, that is for  $f > 0$  with shallow water on the right



Recall the conservation of potential vorticity

$$\frac{\partial}{\partial t} \left( \frac{f + \xi}{h} \right) = 0 \quad q = \frac{f + \xi}{h}$$

that was derived for a more general case  $R_0 \sim R_{0T} \sim O(1)$

$$E_v \ll 1 \quad p' = 0$$

without any reference to the particulars of

$$f \text{ or } h$$

$$\text{Hence will } f = f_0 + \beta_0 y \quad \beta_0 L / f_0 \ll 1$$

$$\text{and } h = H_0 + \alpha_{01} x + \alpha_{02} y \quad \frac{\alpha_{01} L}{H_0} \ll 1$$

The potential vorticity becomes

$$\frac{\partial}{\partial t} q = \frac{f_0 + \beta_0 y + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}{H_0 + \alpha_{01} x + \alpha_{02} y + \eta} = \text{const. following a fluid column}$$

$$\begin{aligned} & \text{provide for pot. vorticity gradients!} \\ & \equiv f_0 + \beta_0 y - \underbrace{\frac{\alpha_{01} f_0}{H_0} x}_{\text{ambient planetary vorticity}} - \underbrace{\frac{\alpha_{02} f_0}{H_0} y}_{\text{bottom}} + \underbrace{\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}_{\text{rel. vorticity}} - \underbrace{\frac{f_0 \eta}{H_0}}_{\text{vortex tube stretching by free surface}} \end{aligned}$$

- If has  $\omega_{max} = \frac{\alpha_0 g}{2 f_0 a}$

- Long waves have high frequency (almost non-dispersive)

- Short waves have low frequency (highly dispersive)