

- slides general circulation (Stommel)
 western boundary current (John Hophis)
 shear instability (Held, 1980)
 movie Gulf Stream (28 frames)

Barotropic Instability

$$u = \bar{U}(y) + u'(x, y, t)$$

$$v = v'(x, y, t)$$

$$p = \underbrace{\bar{P}(y)}_{\text{basic flow mean}} + \underbrace{p'(x, y, t)}_{\text{small perturbations}}$$



$$\bar{U} = \bar{U}(y) \text{ known}$$

and geostrophic

$$+f\bar{U} = -\frac{1}{\rho} \frac{\partial p}{\partial y}$$

$$\bar{\xi}_G = -\frac{d\bar{U}}{dy}$$

nonlinear
dynamics allow

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} \quad \left| - \frac{\partial \psi}{\partial y} \right.$$

transfer of
energy from
one frequency to
another!

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} \quad \left| + \frac{\partial \psi}{\partial x} \right.$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\text{introduce } u' = -\frac{\partial \psi}{\partial y} \quad v' = +\frac{\partial \psi}{\partial x}$$

to satisfy continuity

[Linearize] see p(89a)

↓

continuity equation becomes

$$\left(\frac{\partial}{\partial t} + \bar{U} \frac{\partial}{\partial x} \right) \nabla^2 \psi + \left(\beta_0 \frac{\partial^2 \bar{U}}{\partial y^2} \right) \frac{\partial \psi}{\partial x} = 0$$

$$\beta\text{-effect: } \beta_0 = \frac{df}{dy}$$

relative vorticity gradient $\frac{d\bar{\xi}_G}{dy}$

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}$$

relative

planetary vorticity augmented

$\frac{\partial}{\partial y}$

$$= \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}$$

vorticity of

the perturbation

by the relative vorticity gradient

of the "mean" geostrophic flow \bar{U}

$$= \bar{\xi}'$$

$$v \cdot \frac{d}{dy} (f + \bar{\xi}_G) = \frac{D}{Dt} f + \bar{\xi}_G \quad \text{for } H = \text{const}$$

(89g)

Take the curl to get vorticity equation:

$$\frac{\partial u'}{\partial t} + U \frac{\partial u'}{\partial x} + v' \frac{\partial U}{\partial y} - f v' = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} \quad \left| \frac{\partial}{\partial y} \right.$$

geostrophic basic state

$$\frac{\partial v'}{\partial t} + U \frac{\partial v'}{\partial x} + f u' + f U = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} - \frac{1}{\rho_0} \frac{\partial p}{\partial y} \left| \frac{\partial}{\partial x} \right.$$

$$(-) * \frac{\partial^2 u'}{\partial t \partial y} + \frac{\partial U}{\partial y} \frac{\partial u'}{\partial x} + U \frac{\partial^2 u'}{\partial x \partial y} + v' \frac{\partial^2 U}{\partial y^2} + \frac{\partial U}{\partial y} \frac{\partial v'}{\partial x} - \frac{\partial f}{\partial y} v' - f \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial^2 p}{\partial y^2}$$

$$+ \frac{\partial^2 v'}{\partial t \partial x} + U \frac{\partial^2 v'}{\partial x \partial y} + f \frac{\partial u'}{\partial x} = -\frac{1}{\rho} \frac{\partial^2 p}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial}{\partial t} \left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} \right) + U \left(\frac{\partial^2 v'}{\partial x \partial y} - \frac{\partial^2 u'}{\partial x \partial y} \right) + f \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) \xrightarrow{\nabla^2 \psi}$$

$$+ \frac{\partial U}{\partial y} \left(-\frac{\partial u'}{\partial x} - \frac{\partial v'}{\partial y} \right) + v' \frac{\partial^2 U}{\partial y^2} + \beta_0 v' = 0$$

$$u' = -\frac{\partial \psi}{\partial y} \quad \frac{\partial u'}{\partial y} = -\frac{\partial^2 \psi}{\partial y^2} \quad v' = +\frac{\partial \psi}{\partial x} \quad \frac{\partial v'}{\partial x} = +\frac{\partial^2 \psi}{\partial x^2}$$

$$\frac{\partial u'}{\partial x} = -\frac{\partial^2 \psi}{\partial x \partial y} \quad \frac{\partial v'}{\partial y} = -\frac{\partial^2 \psi}{\partial x \partial y}$$

$$\begin{aligned} \frac{\partial^2 \psi}{\partial x \partial y} &= \frac{\partial^2}{\partial x \partial y} \left(\frac{\partial \psi}{\partial x} \right) \\ -\frac{\partial^2 u'}{\partial x \partial y} &= +\frac{\partial^2}{\partial x \partial y} \frac{\partial \psi}{\partial y} \\ \frac{\partial}{\partial x} \left(\frac{\partial^2 \psi}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x \partial y} \right) &= -\frac{\partial^2 \psi}{\partial x \partial y^2} \end{aligned}$$

(875)

$$\frac{\partial}{\partial t} \left(\nabla^2 \psi \right) + u \frac{\partial}{\partial x} \left(\nabla^2 \psi \right) + \frac{\partial \psi}{\partial x} \left(\beta_0 - \frac{\partial^2 u}{\partial y^2} \right) = 0$$

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) \nabla^2 \psi + \frac{\partial \psi}{\partial x} \left(\beta_0 - \frac{\partial^2 u}{\partial y^2} \right) = 0$$

recall that $\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = \xi'$

relative
vorticity

recall that $\left(\beta_0 - \frac{\partial^2 u}{\partial y^2} \right) \cdot v' = v \frac{\partial}{\partial y} (f + \xi_g)$

because $V = \sigma$ or $v = v'$

hence

$$\xi_g = \frac{\partial V}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial u}{\partial y}$$

geostrophic basic flow is

$$\text{zonal } U = U(y)$$

$$\text{thus } u \frac{\partial}{\partial x} (f + \xi_g) = \sigma$$

and

$$\frac{\partial}{\partial t} (f + \xi_g) = 0$$

Thus

$$\beta_0 - \frac{\partial^2 U}{\partial y^2} = \frac{D}{Dt} \left(\frac{f + \xi_g}{H} \right) \quad \text{for } H = \text{const.}$$

Try solutions

$$\Psi(x, y, t) = \phi(y) e^{i\ell(x-ct)}$$

regular waves with phase speed $c = \omega/\ell$

but what if $c = c_r + i c_i$ is a complex number?

(mathematical proposition)

Then

$$\Psi(x, y, t) = \underbrace{\phi(y) e^{i\ell(x-ct)}}_{\text{regular wave}} \cdot \underbrace{e^{+c_i t}}_{\substack{\text{exponential growth} \\ \text{factor!} \quad \text{if } c_i > 0 \\ \text{growth rate} = c_i \ell}} \quad (\text{physical interpretation})$$

This is exactly what is possible if we introduce boundary conditions

$$\phi(y = -L) = \phi(y = +L) = 0$$

along with the wave solutions that give us a vorticity equation

$$\frac{d^2 \phi}{dy^2} - \left[\ell^2 - \left(\beta_0 - \frac{d^2 U}{dy^2} \right) \middle|_{(U-c)} \right] \phi = 0$$

which allows non-trivial ($\phi \neq 0$) solutions only for a discrete and ordered set of generally complex (phase speeds) eigenvalues c

Think of $c = \frac{\omega}{\ell}$ as a "free" parameter that "forces" only is allowed only if the corresponding waves "fit" the channel
 → EIGENVALUE PROBLEM