

Try solutions

$$\Psi(x, y, t) = \phi(y) e^{i k(x - ct)}$$

regular waves with phase speed  $c = \omega/k$

but what if  $c = c_r + i c_i$  is a complex number?

(mathematical proposition)

Then

$$\Psi(x, y, t) = \underbrace{\phi(y)}_{\text{regular wave}} e^{\underbrace{i k(x - c_r t)}_{+ c_i k t}}$$

$\underbrace{e}_{\text{exponential growth}}$

factor! if  $c_i > 0$

growth rate  $= c_i k$  (physical interpretation)

This is exactly what is possible if we introduce boundary conditions

$$\phi(y=0) = \phi(y=L) = 0$$

along with the wave solutions that give us a continuity equation

$$\frac{d^2 \phi}{dy^2} - \left[ k^2 - \left( \beta_0 - \frac{d^2 U}{dy^2} \right) / (U - c) \right] \phi = 0$$

Rayleigh eq.

which allows non-trivial ( $\phi \neq 0$ ) solutions only for a discrete and ordered set of generally complex (phase speeds) eigenvalues  $c$

Think of  $c = \omega/k$  as a "free" parameter that ~~"forces"~~ only is allowed only if the corresponding waves "fit" the channel

→ EIGENVALUE PROBLEM

Both the eigenvalue  $c$  and eigenfunction  $\phi$  are complex

So if  $c \rightarrow \phi$  then complex conjugate  
 $c^* \rightarrow \phi^*$  are solutions also

A

Flow is stable if and only if  $c$  is real

A

Multiply Rayleigh equation by complex conjugate  $\phi^*$ :  
 and integrate from the boundary to boundary

$$\int_0^L \phi^* \frac{d^2\phi}{dy^2} = \left. \phi^* \frac{d\phi}{dy} \right|_0^L - \int_0^L \frac{d\phi}{dy} \frac{d\phi^*}{dy} dy$$

$\underbrace{\quad}_{\phi(y=0) = \phi^*(y=0)}$

$$\begin{aligned} \phi(y=0) &= \phi^*(y=0) \\ &= \phi^*(y=L) \\ &= \phi(y=L) = 0 \end{aligned}$$

$$= - \int_0^L \left| \frac{d\phi}{dy} \right|^2 dy$$

$$\underbrace{- \int_0^L \left| \frac{d\phi}{dy} \right|^2 dy}_{\text{always real}} + \int_0^L \frac{\beta_0 - \frac{d^2U}{dy^2}}{(U-c)} |\phi|^2 dy = 0$$

$\frac{1}{U-c}$  is complex

$$\frac{1}{U-c} = \frac{(U-c_r) + i c_i}{(U-c_r)^2 + c_i^2} \quad C = C_r + i C_i$$

b

$$\operatorname{Im}\left(\frac{1}{U-c}\right) = c_i / (U-c_r)^2 + c_i^2$$

Thus the imaginary part of the integral becomes

$$c_i \int_0^L \left( \beta_0 - \frac{d^2 U}{dy^2} \right) \cdot \frac{1}{(U-c_r)^2 + c_i^2} \cdot |\phi|^2 dy = 0$$

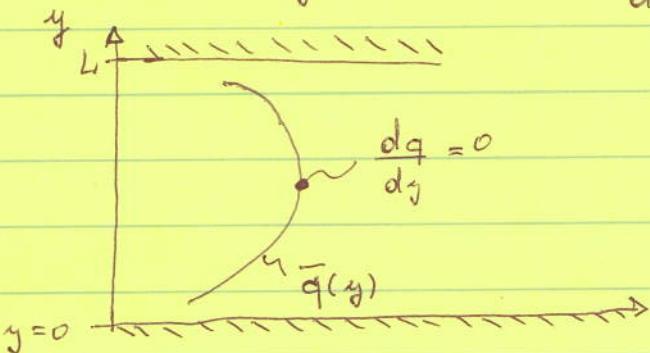
You be both      > 0      > 0  
 > 0 and < 0

Recall that for instability (unlimited, exponential growth)  
we need

$$c_i > 0$$

Hence a necessary (but not sufficient) condition is that

~~$\beta_0 - \frac{d^2 U}{dy^2}$~~  is both positive AND negative somewhere in the domain



$$\bar{q} = f + \bar{\xi}_G = f + \left( \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right)$$

$$\frac{dq}{dy} = \beta_0 - \frac{d^2 U}{dy^2}$$

Potential Vorticity changes sign within the domain

Necessary condition for instability : Potential vorticity gradient vanishes ( $\approx 0$ ) somewhere in domain

Sufficient condition for stability : Potential vorticity gradient does NOT vanish anywhere (always  $\neq 0$ )

Kuo (1949)

So, are Rossby waves stable ?

$$[ f \gg \xi = \frac{\partial \zeta}{\partial x} - \frac{\partial u}{\partial y} ]$$

$$\bar{q} = \frac{f_0 + \beta_0 y}{H} \quad \text{or} \quad \frac{f}{H} = \text{const}$$

$$\Downarrow \frac{dq}{dy} = \frac{\beta_0}{H} = \text{const.}$$

$\Downarrow$  always stable

the  $\beta$ -effect is stabilizing as it has one (positive) sign

$\Downarrow$  So it is the MEAN FLOW + ROSSBY WAVES

that leads to instabilities, that is, the Rossby waves grow without bounds if they are able to extract energy from the mean ambient flow.

The real part of the Rayleigh eq. integral is :

$$-\int_{+0}^L \left| \frac{d\phi}{dy} \right|^2 dy + \int_0^L l^2 |\phi|^2 dy + \int_0^L \underbrace{(\bar{U} - c_r) \left( \beta_o - \frac{d^2 \bar{U}}{dy^2} \right)}_{\frac{d\bar{q}}{dy}} |\phi|^2 \cdot \frac{1}{(\bar{U} - c_r)^2 + c_i^2} dy = 0$$

or  $\int_0^L (\bar{U} - c_r) \frac{d\bar{q}}{dy} |\phi|^2 \cdot \frac{1}{(\bar{U} - c_r)^2 + c_i^2} dy > 0$

$$+ \int_0^L \underbrace{\underbrace{(\bar{c}_r - \mu_o)}_{\text{constant}} \frac{d\bar{q}}{dy} |\phi|^2}_{\text{imaginary part of Rayleigh's integral eq.}} \cdot \frac{1}{(\bar{U} - c_r)^2 + c_i^2} dy$$

this integral vanishes for an instability

$(\bar{c}_r - \mu_o)$  is a constant

$$\int_0^L (\bar{U} - \mu_o) \frac{d\bar{q}}{dy} |\phi|^2 \cdot \frac{1}{(\bar{U} - c_r)^2 + c_i^2} dy > 0$$

always  $> 0$

Thus

$$(\bar{U} - \mu_o) \left( \beta_o - \frac{d^2 \bar{U}}{dy^2} \right) > 0$$

Stronger criteria for  $\mu_o = \bar{U}(y_{\max})$

must be positive for some finite portion of  $y \in [0, L]$  for ANY constant velocity  $\mu_o$  including  $\bar{U}(y_{\max})$  where  $y_{\max}$  is the location that

$$\frac{d\bar{q}}{dy} = 0$$

How fast will perturbations grow?

Howard (1961) restrict ourselves to  $f = \text{const.}$  ( $\beta_0 = 0$ )

1. change of variables from stream function to north-south (meridional) displacement  $\bar{Y}$

$$\bar{v} \equiv v' = \frac{\partial \bar{Y}}{\partial t} + u \frac{\partial \bar{Y}}{\partial x} + v \frac{\partial \bar{Y}}{\partial y} = \frac{D}{Dt} \bar{Y}$$

$$v' = \left[ \frac{\partial \psi}{\partial x} = \frac{\partial \bar{Y}}{\partial t} + U \frac{\partial \bar{Y}}{\partial x} \right] + u' \frac{\partial \bar{Y}}{\partial x} + v' \frac{\partial \bar{Y}}{\partial y}$$

perturbations  $(u', v')$  are small relative to zonal shear flow  $U(y)$

Introduce

$$\psi = \phi(y) e^{il(x-ct)}$$

$$\bar{Y} = a(y) e^{il(x-ct)}$$

And this transformation becomes

$$\boxed{\phi = (U - c) \cdot a}$$

into Rayleigh equation and we get ( $\beta_0 = 0$ )

$$\frac{d}{dy} \left[ (U - c)^2 \frac{da}{dy} \right] - l^2 (U - c)^2 a = 0$$

Same procedure as before (multiply with complex conjugate  $\alpha^*$ , integrate over domain, separate into real and imaginary parts)

$$\text{Real: } \int_0^L [(\bar{U} - c_r)^2 - c_i^2] P dy = 0$$

$$\text{Imaginary: } \int_0^L (\bar{U} - c_r) P dy = 0$$

$$P = \underbrace{\left| \frac{da}{dy} \right|^2 + \ell^2 |a|^2}_{> 0}$$

↓

$\bar{U} - c_r$  must vanish somewhere in  $[0, L]$  where

$$\bar{U} \in [\bar{U}_{\min}, \bar{U}_{\max}]$$

$$\text{or } \bar{U}_{\min} < c_r < \bar{U}_{\max}$$

1 An unstable wave form must travel with a phase speed that matches that of the entraining flow in at least one location

or

There is a place within the domain where the wave does not drift with the ambient flow. This is the location where the wave perturbation extracts energy from the mean flow and grows.

This location where  $\bar{U}(y) = c_r$  is called critical level

Now seek bounds for the imaginary part  $c_i$  as well

$$\text{Real : } \int_0^L [(\bar{U} - c_r)^2 - c_i^2] P dy = 0$$

$$\text{Magick-1: } + \int_0^L (\bar{U} - U_{\min}) (U_{\max} - \bar{U}) P dy \geq 0$$

$$(\bar{U} - U_{\min}) \geq (U - U_{\min}) \cdot \frac{U}{U_{\max}}$$

or

Magick-2  
(Imaginary)

$$I > U/U_{\max}$$

$$- (U_{\min} + U_{\max} - 2c_r) \cdot \int_0^L (U - c_r) \cdot P dy = 0$$

Lots of algebra

$$\Rightarrow \left[ \left( c_r - \frac{U_{\min} + U_{\max}}{2} \right)^2 + c_i^2 - \left( \frac{U_{\max} - U_{\min}}{2} \right)^2 \right] \int_0^L P dy \leq 0$$

$\leq 0$      $\geq 0$

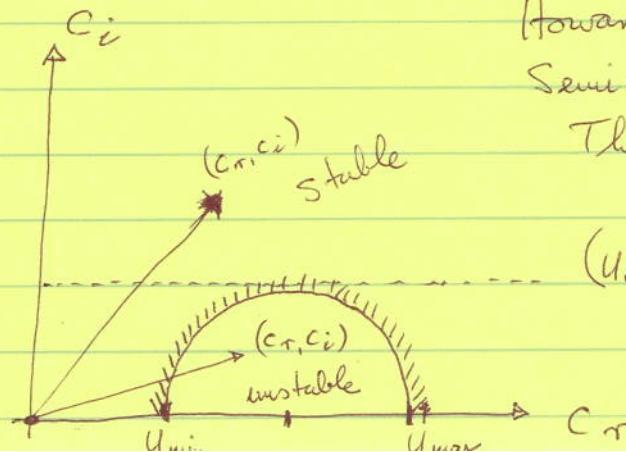
OR

$$\left( c_r - \frac{U_{\min} + U_{\max}}{2} \right)^2 + c_i^2 \leq \left( \frac{U_{\max} - U_{\min}}{2} \right)^2$$

This describes a circle  
in the complex plane :

$$U_{\min} < c_r < U_{\max} \sim u$$

$$c_i \leq \frac{U_{\max} - U_{\min}}{2} \sim \Delta u$$



Howard's  
Semi-circle  
Theorem

$$(U_{\max} - U_{\min})/2$$

## Howard's Semi-circle Theorem

does apply to  $\beta_0 \neq 0$  as well with the

modification

$$U_{\min} - \underbrace{\frac{\beta_0 L^2}{2(\pi^2 + l^2 L^2)}}_{\text{Doppler shift on } \beta\text{-plane}} < c_r < U_{\max}$$

for Rossby waves in channel

$$\left(c_r - \frac{U_{\max} + U_{\min}}{2}\right)^2 + c_i^2 \leq \left(\frac{U_{\max} - U_{\min}}{2}\right)^2 + \frac{\beta_0 L^2 (U_{\max} - U_{\min})}{2(\pi^2 + l^2 L^2)}$$

Pedlosky (1987)  
Section 7.5