

$$\frac{du}{dt} - f_0 v - \beta_0 y v = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \gamma \frac{\partial^2 u}{\partial x^2}$$

$$\frac{dv}{dt} + f_0 u - \beta_0 y u = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + \gamma \frac{\partial^2 v}{\partial y^2}$$

$$\sigma = -\frac{\partial p}{\partial z} - \rho' g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \sigma$$

$$\frac{\partial p'}{\partial t} + u \frac{\partial p'}{\partial x} + v \frac{\partial p'}{\partial y} + w \frac{\partial \bar{p}}{\partial z} = \sigma$$

With $R_{0T} \sim Fr^2 \sim E_v \sim \beta \sim R_0 \ll 1$

[we will return to this later again]

and $u = u_0 + u_1 + u_2 + \dots$

$$u = U \left[u_0^* + R_0 u_1^* + \sigma(R_0^2) \right]$$

$$u_0 = U \cdot u_0^* = O(U)$$

$$p = P \left[p_0^* + R_0 p_1^* + \sigma(R_0^2) \right]$$

$$u_1 = U \cdot u_1^* = O(R_0 U)$$

The $O(1)$ equations are geostrophic

$$v_0 = \frac{1}{\rho_0 f_0} \frac{\partial p_0}{\partial x}, \quad u_0 = -\frac{1}{\rho_0 f_0} \frac{\partial p_0}{\partial y}$$

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = \sigma = \frac{\partial w_0}{\partial z}$$

Then the full geostrophic x-momentum written in terms of geostrophic velocity or pressure becomes

$$\underbrace{-\frac{1}{\rho_0 f_0} \frac{\partial}{\partial y} \frac{\partial p_0}{\partial t}}_{O(R_0 \tau)} - \underbrace{\frac{1}{\rho_0 f_0^2} \mathcal{J}\left(\rho_0, \frac{\partial \rho_0}{\partial y}\right)}_{O(R_0)} - \underbrace{\frac{1}{\rho_0 f_0} w \frac{\partial^2 \rho_0}{\partial y \partial z}}_{O(R_0 \cdot Fr^2)} - \boxed{f_0 v_1} - \underbrace{\frac{\beta_0}{\rho_0 f_0} y \frac{\partial \rho_0}{\partial y}}_{O(\beta)} =$$

from $\frac{\partial u}{\partial t}$
 $u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}$ from
 $w \frac{\partial u}{\partial z}$

$$= \underbrace{-\frac{1}{\rho_0} \frac{\partial p_1}{\partial x}}_{O(R_0)} - \underbrace{\frac{v}{\rho_0 f_0} \frac{\partial^3 \rho_0}{\partial y \partial z^2}}_{O(E_v)}$$

$$\mathcal{J}(a, b) = \frac{\partial a}{\partial x} \frac{\partial b}{\partial y} - \frac{\partial a}{\partial y} \frac{\partial b}{\partial x} = -\mathcal{J}(b, a)$$

e.g.:

$$\vec{u}_H \cdot \nabla_H = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}$$

as in

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{(\rho_0 f_0)^2} \mathcal{J}\left(\rho_0, \frac{\partial \rho_0}{\partial y}\right)$$

Or @ $O(R_0, \beta, E_v)$:

$$v_1 = \underbrace{+\frac{1}{\rho_0 f_0^2} \frac{\partial p_1}{\partial x}}_{\text{unknown velocity } (E_v)} - \underbrace{\frac{1}{\rho_0 f_0^2} \frac{\partial^2 \rho_0}{\partial y \partial t}}_{\text{unknown pressure gradient}} - \dots$$

known geostrophic contributions that are ALL known if p_0 is known

$$u_1 = -\frac{1}{\rho_0 f_0^2} \frac{\partial p_1}{\partial y} - \dots$$

Put these (u_1, v_1) into continuity

$$-\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) = \frac{\partial w_1}{\partial z} = + \frac{1}{\rho_0 f_0^2} \left[\frac{\partial}{\partial t} \nabla^2 p_0 + \mathcal{J}(p_0, \nabla^2 p_0) + \beta_0 \frac{\partial p_0}{\partial x} - \nu \nabla^2 \left(\frac{\partial^2 p_0}{\partial z^2} \right) \right]$$

The $\frac{\partial p_1}{\partial x}$ and $\frac{\partial p_1}{\partial y}$ terms dropped out, as we basically took $\frac{\partial}{\partial x} (y\text{-mean}) - \frac{\partial}{\partial y} (x\text{-mean})$

that is, we can interpret this as a vorticity equation that includes a new variable, vertical velocity w_1 in addition to the also an ageostrophic

unknown geostrophic pressure p_0 .

→ need another equation to close the problem:

$$\frac{\partial p_1'}{\partial t} + \frac{1}{\rho_0 f_0} \mathcal{J}(p_0, p_1') - \frac{\rho_0 N^2}{g} w_1 = 0$$

$$= u_0 \frac{\partial p_1'}{\partial x} + v_0 \frac{\partial p_1'}{\partial y} + w_1 \frac{\partial \bar{p}}{\partial z}$$

$$N^2 = -\frac{g}{\rho_0} \frac{\partial \bar{\rho}}{\partial z} = N^2(z)$$

known function of z
background stratification

$$\sigma \quad p_1' = -\frac{1}{g} \frac{\partial p_0}{\partial z}$$

$$w_1 = \frac{g}{N^2 \rho_0} \left[-\frac{1}{g} \frac{\partial}{\partial t} \left(\frac{\partial p_0}{\partial z} \right) + \frac{1}{\rho_0 f_0} \mathcal{J} \left(p_0, -\frac{1}{g} \frac{\partial p_0}{\partial z} \right) \right]$$

Recall

also that

$$p_0 = \rho_0 f_0 \psi$$

where

$$u_0 = -\frac{\partial \psi}{\partial y}, \quad v_0 = +\frac{\partial \psi}{\partial x}$$

$$\downarrow \quad p_1' = -\rho_0 f_0 \frac{1}{g} \frac{\partial p_0}{\partial z} = -\frac{\rho_0 f_0}{g} \frac{\partial \psi}{\partial z}$$

We now have

w_1 , expressed in terms of geostrophic pressure p_0
from the density equation

We also have $\frac{\partial w_1}{\partial z}$ expressed in terms of geostrophic pressure p_0
from the vorticity equation

$$\left. \frac{\partial w_1}{\partial z} \right|_{\text{density equation}} = \left. \frac{\partial w_1}{\partial z} \right|_{\text{vorticity equation}} \quad \text{gives (see p. 117, 118B)}$$

$$\frac{\partial}{\partial t} \left[\nabla^2 p_0 + \frac{\partial}{\partial z} \left(\frac{f_0^2}{N^2} \frac{\partial p_0}{\partial z} \right) \right] + \frac{1}{\rho_0 f_0} \left[\left(\rho_0, \nabla^2 p_0 + \frac{f_0}{N^2} \frac{\partial p_0}{\partial z} \right) + \beta_0 \frac{\partial p_0}{\partial x} \right] = \nabla^2 \frac{\partial^2}{\partial z^2} (\nabla^2 p_0)$$

Single PDE for a continuously stratified fluid under the QG assumption that isopycnal excursions from a basic state are small, that is
 $\rho = \bar{\rho}(z) + \rho'$ where $\rho' \ll \bar{\rho}$

$$\text{Recall: } \frac{\partial p_0}{\partial x} = \rho_0 f_0 v_0, \quad \frac{\partial p_0}{\partial y} = -\rho_0 f_0 u_0, \quad \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = \sigma = -\frac{\partial w_0}{\partial z}$$

$$u_0 = -\frac{\partial \psi}{\partial y}, \quad v_0 = +\frac{\partial \psi}{\partial x} \quad \rightarrow p_0 = \rho_0 f_0 \psi$$

$$\left(\rightarrow \frac{\partial p_0}{\partial z} = \rho_0 f_0 \frac{\partial \psi}{\partial z} = -g \rho' \right)$$

$$q = \nabla^2 \psi + \frac{\partial}{\partial z} \left(\frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \right) + \beta_0 y \quad \text{potential vorticity}$$

$$\frac{\partial q}{\partial t} + J(\psi, q) = \nu \frac{\partial^2}{\partial z^2} (\nabla^2 \psi)$$

$$\frac{D_0}{Dt} (q) = \nu \frac{\partial^2}{\partial z^2} (\nabla^2 \psi)$$

where $\frac{D_0}{Dt} = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} + v_0 \frac{\partial}{\partial y}$

$$\frac{D_0}{Dt} \left[\nabla^2 \psi + \beta_0 y + \frac{\partial}{\partial z} \left(\frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \right) \right] = \nu \frac{\partial^2}{\partial z^2} (\nabla^2 \psi)$$

relative vorticity of the geostrophic flow
vortex tube stretching due to stratification
diffusion of relative vorticity of the geostrophic flow

Observations:

1. How to reconcile with $\frac{D}{Dt} \left(\frac{f + \xi}{h} \right) = 0$?

$\bar{h} \gg h'$

For each "layer" of thickness $h = \bar{h} + h' = h' + \frac{\Delta p}{\partial \bar{p} / \partial z}$

$$q = \frac{f + \xi}{h} = \frac{f_0 + \Delta f + \xi}{\bar{h} + h'} \approx \frac{1}{h} \left(f_0 + \Delta f + \xi - \frac{f_0}{h} h' + \dots \right)$$

$$= q_0 + \frac{q'}{h}$$

where $q_0 = \frac{f_0}{h}$ and $q' = \Delta f + \xi - \frac{f_0}{h} h' = PVAG$

$\underbrace{\hspace{10em}}_{\beta y \nabla^2 \psi} \underbrace{\hspace{10em}}_{\frac{f_0^2}{N^2} \frac{\partial^2 \psi}{\partial z^2}}$