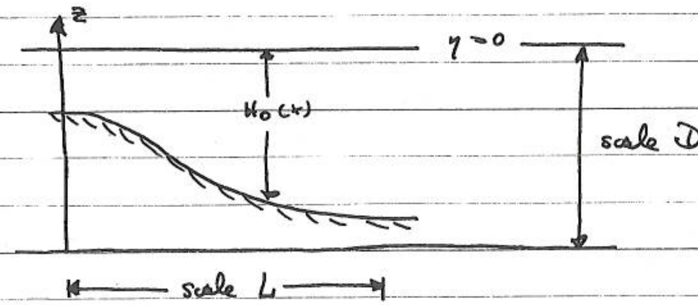
Formulation

linearized inviscid shallow water equations with topography as



$$\frac{\partial u}{\partial t} - f v = -g \frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + f u = -g \frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(u H_0) + \frac{\partial}{\partial y}(v H_0) = 0$$

Layer thickness $H(x,y,t) = H_0(x,y) + \eta(x,y,t)$, H_0 layer thickness for the fluid at rest

restrictions : wave frequencies $\sigma \ll f = \text{const.}$ (f-plane)

$kL \ll 1$ (long waves ~~only~~ } along y only)

Pedlosky (1979), Chapter 8

$F \equiv (L/R)^2 = \left[L / (\sqrt{gD}/f)\right]^2 \ll 1$ Rossby radius of deformation is much bigger than the applied lengthscale in the x-direction

(1) The conservation of mass in an incompressible fluid is stated as

$$\nabla \cdot \vec{u} = 0$$

or $$\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

This gives an upper bound for the vertical velocity scale W:

$$\frac{W}{D} \leq \left[\frac{U}{L}, kV\right]_{max}$$

where U, V are ~~horizontal~~ meridional and zonal velocity scales according to their lengthscales L and ~~$2\pi/k$~~ k^{-1}

As long as nothing is specified about the magnitude of W – which after depth integration leads to ~~the surface~~ time rate of) changes of elevation $\partial_t \eta$ – a mass balance is certainly ossured by requiring a vanishing horizontal divergence, that is that the scales

$$\frac{U}{L} \quad \text{and} \quad kV$$

are of the same order. ~~This then allows in the limit of vanishing surface elevation in the ma~~

~~This then allows for very small vertical velocity elevation surface elevation to depth ratio $\eta_0/D \sim T\,W/D$ which could be argued to be small because of linearity.~~

Then

$$U = kL\,V$$

If the horizontal does not vanish, a contribution from $\frac{W}{D}$ enters in the mass balance, then it can be assumed that $W/D = O[U/L, kV]_{max}$

ok

2/2

(2)

Scaling the linearized x-momentum equation

$$\frac{\partial u^*}{\partial t^*} - f v^* = -g \frac{\partial \eta^*}{\partial x^*}$$

$u^* = u \cdot U = u\, kL\, V$, $v^* = v V$

$\frac{\partial}{\partial x^*} = \frac{1}{L}\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y^*} = k \frac{\partial}{\partial y}$

$\frac{\partial}{\partial t^*} = \sigma \frac{\partial}{\partial t^*}$, $\eta^* = a\, \eta$

then

$$\sigma kLV \frac{\partial u}{\partial t} - fV v = -\frac{g a}{L}\frac{\partial \eta}{\partial x}$$

OK, but no need for scaled variables here, just scaling analysis.

✓ or

$$\frac{\sigma}{f}\frac{\partial u}{\partial t} - (kL)^{-1} v = -\frac{g}{fVkL^2}\, a \frac{\partial \eta}{\partial x}$$

$\rightarrow$ ~~…~~ $(\sigma/f) * (kL) \frac{\partial u}{\partial t}$ ✓ $- v = \frac{-g}{fVL}\, a \frac{\partial \eta}{\partial x}$

~~or $(kL)(\sigma/f) \frac{\partial u}{\partial t} - v = \frac{fVL}{g}$~~

All? dimensionaless parameters have to be of the same order for the prescribed x-momentum to hold

$$\rightarrow \quad a = \frac{fV(kL)}{g} L \left[\frac{\sigma}{f}, (kL)^{-1}\right] = \cancel{\frac{fVL}{g}\left[\left(\frac{\sigma/f}{kL}\right)^{-1}, 1\right]} = \frac{fVL}{g}\left[(kL)\frac{\sigma}{f}, 1\right]$$

Therefore an appropriate scale for the surface elevation η^* is

fVL/g ~~if $(\sigma/f) * (kL)$ is $O(1)$~~

6/6

If $kL \ll 1$ and $\sigma < f$ then ✓ the resulting x-momentum with the scale for η chosen as ✓ fVL/g is: $f v = g \partial_x \eta$, geostrophy ✓

(3) (4)

simplified x - momentum

$$-fv = -g\frac{\partial \eta}{\partial x} \quad \Big| \frac{\partial}{\partial y}$$

y - momentum $$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \quad \Big| \frac{\partial}{\partial x}$$

$$\to \quad -f\frac{\partial v}{\partial y} = -g\frac{\partial^2 \eta}{\partial x \partial y}$$

$$\to \quad \frac{\partial^2 v}{\partial t \partial x} + f\frac{\partial u}{\partial x} = -g\frac{\partial^2 \eta}{\partial y \partial x}$$

semi geostrophic

$$\to \quad -f\frac{\partial v}{\partial y} - \frac{\partial^2 v}{\partial t \partial x} - f\frac{\partial u}{\partial x} = 0$$

(1) $$\to \quad -f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) = 0$$

continuity $$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} u H_0(x) + H_0(x)\frac{\partial v}{\partial y} = 0$$

$$\to \quad \frac{\partial \eta}{\partial t} + u\frac{\partial H_0}{\partial x} + H_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

$$\to \quad -\nabla_H \vec{u} = \frac{1}{H_0}\frac{\partial \eta}{\partial t} + \frac{u}{H_0}\frac{\partial H_0}{\partial x}$$

in (1) :

$$\cancel{f}\left(\frac{1}{H_0}\frac{\partial \eta}{\partial t} + \frac{u}{H_0}\frac{\partial H_0}{\partial x}\right) - \frac{1}{f}\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) = 0 \quad \text{or} \quad \frac{\partial}{\partial t}\left(\frac{\eta}{H_0} - \frac{1}{f}\frac{\partial v}{\partial x}\right) + \frac{u}{H_0}\frac{\partial H_0}{\partial x} = 0$$

vorticity

$$\frac{1}{H_0}\frac{\partial \eta}{\partial t} + \frac{u}{H_0}\frac{\partial H_0}{\partial x} - \frac{1}{f}\frac{\partial}{\partial t}\frac{\partial v}{\partial x} = 0$$

or $\frac{\partial}{\partial t}\left(\frac{\eta}{H_0} - \frac{1}{f}\frac{\partial v}{\partial x}\right) + \frac{u}{H_0}\frac{\partial H_0}{\partial x} = 0$

$\Rightarrow \frac{\partial}{\partial t}\left(\frac{f\eta - H_0 \partial_x v}{f \cancel{H_0}}\right) + \frac{1}{\cancel{H_0}}\frac{\partial H_0}{\partial x} = 0$

$\Rightarrow \partial_t (f\eta - H_0 \partial_x v) + f u \, \partial_x H_0 = 0$

$\partial_t (f\eta - H_0 \partial_x v) + f u \, \partial_x H_0 = 0$

but from momentum $v = \frac{-g}{f}\frac{\partial \eta}{\partial x}$ $\Rightarrow$ $\frac{\partial v}{\partial x} = -\frac{g}{f}\frac{\partial^2 \eta}{\partial x^2}$

and $u = -\frac{g}{f}\frac{\partial \eta}{\partial y} - \frac{1}{f}\frac{\partial v}{\partial t} = -\frac{g}{f}\frac{\partial \eta}{\partial y} + \frac{g}{f^2}\frac{\partial^2 \eta}{\partial x \partial t}$

$$\Rightarrow \frac{f}{g}\frac{1}{H_0}\frac{\partial \eta}{\partial t} + \left((-)\frac{\cancel{g}}{\cancel{f}}\frac{\partial \eta}{\partial y} + \frac{\cancel{g}}{f^{\cancel{2}}}\frac{\partial^2 \eta}{\partial x \partial t} \right) \cdot \frac{\partial H_0}{\partial x} - \frac{1}{f}\frac{\partial}{\partial t}\left[(-)\frac{\cancel{g}}{\cancel{f}}\frac{\partial^2 \eta}{\partial x^2} \right] = 0 \qquad \Big/ \cdot \frac{f}{g}$$

6/7

$$\Rightarrow \cancel{\frac{\partial}{\partial t}} \; \frac{f^2}{g H_0}\frac{\partial \eta}{\partial t} + \left(-\frac{\partial \eta}{\partial y} + \frac{\partial^2 \eta}{\partial x \partial t} \right) f \cdot \frac{\partial H_0}{\partial x} + \cancel{\frac{1}{f}}\frac{\partial}{\partial t}\frac{\partial^2 \eta}{\partial x^2} = 0$$

Cons. pot. vort. $\leftrightarrow$ Cons. ang. mom.

$$\Rightarrow \frac{\partial}{\partial t}\left[\frac{f^2}{g H_0}\eta + \frac{f \partial H_0}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 \eta}{\partial x^2} \right] - f \frac{\partial \eta}{\partial y}\frac{\partial H_0}{\partial x} = 0$$

~~$\frac{\partial \eta}{\partial t}$ + $f u$ = $-g\frac{\partial \eta}{\partial y}$ ... η ... mom.~~

$$\Rightarrow \frac{\partial}{\partial t}\left[\underbrace{\frac{\partial^2 \eta}{\partial x^2}}_{\xi} + \underbrace{\frac{f^2}{g H_0}\eta}_{\eta / R^2} + \underbrace{f \frac{\partial H_0}{\partial x}\frac{\partial \eta}{\partial x}}_{\text{ambient vorticity}} \right] - \underbrace{f \frac{\partial \eta}{\partial y}\frac{\partial H}{\partial x}}_{\text{stretching due to topography}} = 0$$

? This is a statement about conservation of angular momentum. It is not a statement of pot. vorticity conservation.

Scaling verification

(4) $$\frac{\partial}{\partial t^*}\left(f\eta^* - H_0^* \frac{\partial v^*}{\partial x^*}\right) + f u^* \frac{\partial H_0^*}{\partial x^*} = 0$$

scaling with $x^* = Lx$, $H_0^* = h_0 D$, $u^* = kLVu$, $v^* = Vv$

$$\eta^* = \frac{fVL}{g}, \quad \frac{\partial}{\partial t^*} = \sigma\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x^*} = \frac{1}{L}\frac{\partial}{\partial x}$$

$\Rightarrow$

$$\sigma\frac{\partial}{\partial t}\left(\cancel{f}\cdot\frac{fVL}{g\cancel{D}}\eta - \frac{h_0\cancel{D}}{fL} V \frac{\partial v}{\partial x}\right) + \cancel{f}\cdot\frac{kLV\cancel{D}}{L} u \frac{\partial h_0}{\partial x} = 0 \quad \Big| : fD$$

$$\frac{\partial}{\partial t}\left(\frac{\sigma f VL}{gD}\eta - \frac{\sigma V}{fL} h_0\frac{\partial v}{\partial x}\right) + kV u\frac{\partial h_0}{\partial x} = 0 \quad \Big| \cdot L/V$$

$$\frac{\partial}{\partial t}\left(\frac{\sigma}{f}\frac{f^2}{gD}\cancel{V}L^2\eta - \frac{\sigma}{f}\frac{\cancel{V}}{\cancel{L}} h_0\frac{\partial v}{\partial x}\right) + k\cancel{V}L u\frac{\partial h_0}{\partial x} = 0$$

$\Rightarrow$

$$\frac{\partial}{\partial t}\left(\frac{\sigma}{f}F\eta - \frac{\sigma}{f} h_0\frac{\partial v}{\partial x}\right) + kL u\frac{\partial h_0}{\partial x} = 0$$

(a) $$\Rightarrow \frac{\sigma}{f}\frac{\partial}{\partial t}\left(F\eta - h_0\frac{\partial v}{\partial x}\right) + kL u\frac{\partial h_0}{\partial x} = 0$$ ✓

(4b) $$\frac{\nabla}{f}\frac{\partial}{\partial t}\left(F\eta - h_0\frac{\partial v}{\partial x}\right) + kL\, u\frac{\partial h_0}{\partial x} = 0$$

~~for F≪1~~ I can't get the sign right

(4c) for this vorticity balance to hold $\left(\frac{\nabla}{f}\right) = \left(kL\right)$ long wave, short period frequency

$\frac{\nabla}{f}$ $\begin{cases} F\eta \text{ is the vortex stretching of the free surface} \\ h_0\partial_x v \text{ is the relative vorticity because of the geostrophic x-momentum } v \propto \partial_x \eta \end{cases}$

$kL\, u\frac{\partial h_0}{\partial x}$ is a vortex stretching contribution (forcing) due to the sloping bottom.

~~$c = \frac{\nabla(k)}{k} = +Lf$ non dispersive, $c = c_{gr}$~~

10/10

For $F \ll 1$ the vorticity can be written as

$$-\frac{\nabla}{f}\frac{\partial}{\partial t}\left(h_0\frac{\partial v}{\partial x}\right) + kL\, u\frac{\partial h_0}{\partial x} = 0$$

? x, y orientation arb. here

minus sign can't be obtained by scaling analysis

4d) an estimate of the meridional phase velocity $C \approx \frac{\nabla(k)}{k} \approx -fL$ where $\nabla(k) \sim -fkL$, that is the southward propagating wave is non-dispersive $C = C_{gr}$

(5) y-momentum (same scaling as for the x-momentum)

$$\frac{\partial v^*}{\partial t^*} - f u^* = -g \frac{\partial \eta^*}{\partial y^*}$$

$$\sigma \cancel{V} \frac{\partial v}{\partial t} - f\, kL\cancel{V}\, u = -\cancel{g}\, k \frac{f \cancel{V} L}{\cancel{g}} \frac{\partial \eta}{\partial y} \qquad \Big| : f$$

$$\frac{\sigma}{f} \frac{\partial v}{\partial t} - kL\, u = -kL \frac{\partial \eta}{\partial y}$$

$$\frac{\sigma/f}{kL} \frac{\partial v}{\partial t} - u = -\frac{\partial \eta}{\partial y} \checkmark$$

4/4

To stay consistent with the scaling and the approximation ~~done~~ applied to the vorticity equation, i.e.: $\sigma/f \approx kL$, ~~the~~ all terms in the y-momentum are of $\mathcal{O}(1)$. ✓

Then the ~~total~~ balance is geostrophic in x and ageostrophic in y, because of the term $\partial_t v$. The main point here is that all terms are of the same order. No expansion in terms of small Rossby # has been undertaken.

To my present ability it is not possible to write the vorticity equation in a conservation law form as it is done in eq. (3.12.25). It is

There is ~~an~~ vorticity production term $f \frac{\partial y}{\partial y} \frac{\partial H}{\partial x}$ which prevents this.

just the stretching term.

The reason is that the assumed lowest order ~~balance~~ momentum Cons. balance is not geostrophic in x and y. It is geostrophic only in x, but already time dependent (not degenerated) and therefore ageostrophic in the y momentum.

pot. vort. operates even in non-rot. flow, provided no torques, as here.

In the presence of boundaries the equation (3.12.25) allows quite general dispersive waves whose phase propagates always to the West. In our example we have explicitly assumed ~~waves~~ long waves in the y-direction. This wave is in geostrophic balance ~~along isobath~~ in the direction ~~along the isobath~~ of its phase propagation which happens to be along isobath.

?

The bal. that is geosth. is along $x \perp \vec{c}$.

But exactly this particle velocity along isobaths is a forcing velocity for the ageostrophic cross-isobath flow (as governed by the y-momentum) <u>and</u> this ageostrophy is the ~~only~~ driving term for the wave, because it is the only time dependence in the linear momentum equation.

A little confusion is on my side because I do not quite know what the order of the considered dynamics is. If they are of $O(1)$ as they appear to be, if the problem is not degenerated all three equations (vorticity or continuity and the linear momentum equations) have to be solved simultaneously.

(6) The linear quasi-geostrophic Rossby waves discussed by Pedlosky (1979, ch. 3) are based on the assumption that the ambient vorticity of the topography is of the same order as the Rossby # which is assumed to be small already. ✓

In the present exercise no a priori assumption about the magnitude of the depth variation ~~to~~ relative to the total depth scale are made. This main difference is expressed best by comparing the two vorticity equations. ✓

Pedlosky (1979, eq. 3.12.25) :

Here $u/v \ll 1$.

$$(3.12.25) \qquad \frac{d}{dt}\left(\xi - F\eta_0 + \eta_B\right) = 0$$

4/6

where

$\xi = \nabla_H^2 \eta_0$ relative vorticity

$F\eta_0 = (L/R)^2 \eta_0$ is vortex stretching due to the free surface, R being the rossby radius of deformation

η_B is the ambient vorticity

This statement is one about the conservation of potential vorticity, understood as following a material vortex advected with the $O(1)$ geostrophic velocity field. ✓

In the present exercise we have instead a vorticity equation ~~of~~ as

$$\frac{\partial}{\partial t}\left(\xi - F\eta_0 + \eta_B\right) - f\,\frac{\partial \eta}{\partial y}\,\frac{\partial H}{\partial x} = 0$$

$\xi = \partial_{xx}^2 \eta_0$ $\qquad \eta_B = f\,\frac{\partial H_0}{\partial x}\,\frac{\partial \eta_0}{\partial x}$

$F\eta_0 \propto \left(\frac{L}{R}\right)^2 \eta_0$

from page 5