

Appendix A

Wave Kinematics

Summary: Because numerous geophysical flow phenomena can be interpreted as waves, some understanding of basic wave properties is required in the study of geophysical fluid dynamics. The concepts of wave number, frequency, dispersion relation, phase speed, and group velocity are introduced and given physical interpretations.

A-1 WAVE NUMBER AND WAVELENGTH

For simplicity of presentation and easier graphical representation, we will consider here a two-dimensional wave, namely, a physical signal occupying the (x, y) plane and evolving with time t . The prototypical wave form is the sinusoidal function, and so we assume that a physical variable of the system, denoted by a and being pressure, a velocity component or whatever, evolves according to

$$a = A \cos(lx + my - \omega t + \phi). \quad (\text{A-1})$$

The coefficient A is the wave *amplitude* ($-A \leq a \leq +A$), whereas the argument

$$\alpha = lx + my - \omega t + \phi \quad (\text{A-2})$$

is called the *phase*. The latter consists of terms that vary with each independent variable

and a constant ϕ , called the reference phase. The coefficients l , m , and ω of x , y , and t , respectively, bear the names of *wave number in x* , *wave number in y* , and *frequency*. They indicate how sharply the wave undulates in space and how fast it oscillates in time.

Equivalent expressions for the wave signal are

$$a = A_1 \cos(lx + my - \omega t) + A_2 \sin(lx + my - \omega t), \tag{A-3}$$

where $A_1 = A \cos \phi$ and $A_2 = -A \sin \phi$, and

$$a = \text{Re}[A_c e^{i(lx + my - \omega t)}], \tag{A-4}$$

where the notation $\text{Re}[\]$ means the real part of and $A_c = A_1 - iA_2 = e^{i\phi}$ is a complex amplitude coefficient. The choice of mathematical representation is generally dictated by the problem at hand. Formulation (A-3) is helpful in the discussion of problems exhibiting coexisting signals that are either in perfect phase or in quadrature, whereas formulation (A-4) is preferred when a given dynamical system is subjected to a wave analysis. Here, we will use formulation (A-1).

A wave *crest* is defined as the line in the (x, y) plane and at time t along which the signal is maximum ($a = +A$); similarly, a *trough* is a line along which the signal is minimum ($a = -A$). These lines and, in general, all lines along which the signal has a constant value at an instant in time are called *phase lines*. Figure A-1 depicts a few phase lines in the case of positive wave numbers l and m .

Because of the oscillatory behavior of the sinusoidal function, crest lines recur at constant intervals, thus giving the wavy aspect to the signal. The distance over which

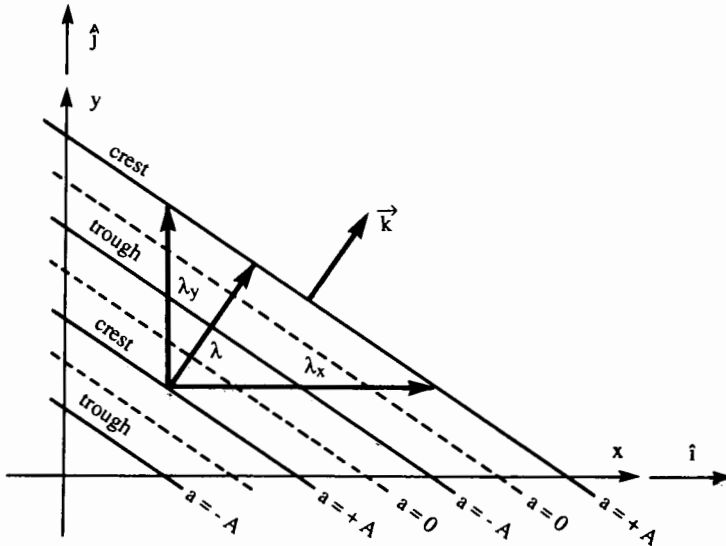


Figure A-1 Phase lines of a two-dimensional wave signal at a given time. The lines are all straight and parallel. The distances from crest to nearest crest along the x - and y -axes are λ_x and λ_y , respectively. The wavelength λ is the shortest diagonal distance from crest to nearest crest.

the signal repeats itself in the x -direction is the distance over which the phase portion lx increases by 2π , that is,

$$\lambda_x = \frac{2\pi}{l}. \quad (\text{A-5})$$

Similarly, the distance over which the signal repeats itself in the y -direction is

$$\lambda_y = \frac{2\pi}{m}. \quad (\text{A-6})$$

The quantities λ_x and λ_y are called the wavelengths in the x - and y -directions. They are the wavelengths seen by an observer who would detect the signal only through slits aligned with the x - and y -axes. The actual *wavelength*, λ , of the wave is the shortest distance from the crest to nearest crest (Figure A-1) and is, therefore, smaller than either λ_x or λ_y . Elementary geometric considerations provide

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_x^2} + \frac{1}{\lambda_y^2} = \frac{l^2 + m^2}{4\pi^2}$$

or

$$\lambda = \frac{2\pi}{k}, \quad (\text{A-7})$$

where k , called the *wave number*, is defined as

$$k = \sqrt{l^2 + m^2}. \quad (\text{A-8})$$

Note that since λ^2 is not the sum of λ_x^2 and λ_y^2 , the pair (λ_x, λ_y) does not represent a vector. On the other hand, the pair (l, m) can be used to define the *wave number vector*

$$\mathbf{k} = l\mathbf{i} + m\mathbf{j}, \quad (\text{A-9})$$

where \mathbf{i} and \mathbf{j} are the unit vectors aligned with the axes (Figure A-1). In this fashion, the wave number k is the magnitude of the wave number vector.

By definition, phase lines at any given time correspond to lines of constant values of the expression $lx + my = \mathbf{k} \cdot \mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ is the vector position. Geometrically, this means that a phase line is the locus of points whose vectors from the origin share the same projection onto the wave-number vector. These points form a straight line perpendicular to \mathbf{k} , and therefore the wave-number vector points perpendicularly to all phase lines (Figure A-1), that is, in the direction of the undulation.

A-2 FREQUENCY, PHASE SPEED, AND DISPERSION

Let us now turn our attention to the temporal evolution of the wave signal. At a fixed position (x and y given), an observer sees an oscillatory signal. The interval of time between two consecutive instants at which the signal is maximum is the time taken for

the portion ωt of the phase to increase by 2π . It is called the *period*, which is

$$T = \frac{2\pi}{\omega}. \tag{A-10}$$

Let us now follow a particular crest line ($a = A$) from a certain time t_1 to a later time t_2 and note the time interval $\Delta t = t_2 - t_1$. During this time interval, the wave crest has progressed from one position to another (Figure A-2). The intersection with the x -axis

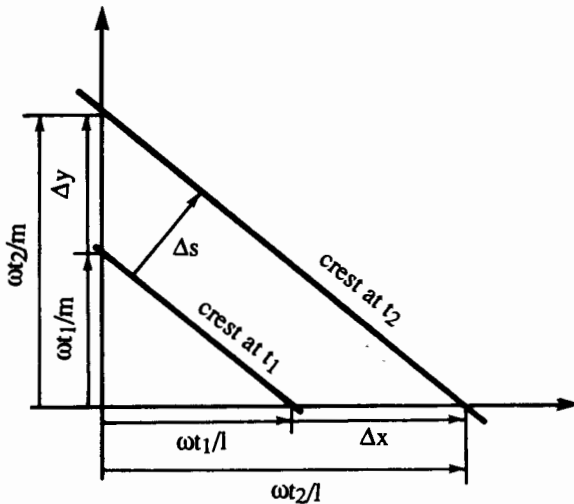


Figure A-2 Progress of a wave crest from time t_1 to time t_2 . The ratio of the distance traveled, Δs , to the time interval $\Delta t = t_2 - t_1$ is the phase speed.

has translated over the distance $\Delta x = \omega t_2/l - \omega t_1/l = \omega \Delta t/l$ in time Δt . This defines the propagation speed of the wave along the x -direction:

$$c_x = \frac{\Delta x}{\Delta t} = \frac{\omega}{l}. \tag{A-11}$$

Similarly, the propagation speed along the y -direction is the distance $\Delta y = \omega t_2/m - \omega t_1/m$ divided by the time interval Δt , or

$$c_y = \frac{\Delta y}{\Delta t} = \frac{\omega}{m}. \tag{A-12}$$

But these speeds are only apparent speeds. The true propagation speed of the wave is the distance Δs , measured perpendicularly to the crest line (Figure A-2), covered by this crest line during the time interval Δt . Again, elementary geometric considerations provide

$$\frac{1}{\Delta s^2} = \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2},$$

from which we deduce

$$\Delta s = \frac{\omega \Delta t}{k},$$

where k is the wave number defined in (A-8). The propagation speed of the crest line is thus

$$c = \frac{\Delta s}{\Delta t} = \frac{\omega}{k}. \quad (\text{A-13})$$

Because all phase lines propagate at the same speed (so that the wave preserves its sinusoidal form), the quantity c is called the *phase speed*. Note that because c^2 is not equal to $c_x^2 + c_y^2$ (in fact, c is less than either c_x and c_y !), the pair (c_x, c_y) does not constitute a physical vector. The direction of phase propagation, as discussed before, is parallel to the wave-number vector \mathbf{k} .

In general, the expression (A-1) of the wave signal appears as the solution to a particular dynamical system. Therefore, it must somehow be constrained by the physics of the problem, and not all its parameters can be varied independently. Let us suppose that the system under consideration is initially unchanging in time (state of rest or steady flow) and that at time $t = 0$, it is perturbed spatially according to a sinusoidal distribution of wave numbers l and m in the x - and y -directions, respectively, and of amplitude A for the variable a . Intuition leads us to anticipate that subsequent to this perturbation, the system will react in a time-dependent fashion. If this reaction takes the form of a wave, it will have a frequency ω determined by the system. Therefore, the frequency can be viewed as dependent upon the wave-number components l and m and the amplitude A . In most instances, the system's response is a wave because the set of equations representing the physics is linear, and when this is the case, the mathematical analysis yields a frequency that is independent of the amplitude of the perturbation. Therefore, ω is typically a function of l and m only.

If the frequency is a function of the wave-number components, so is the phase speed:

$$c = \frac{\omega(l, m)}{\sqrt{l^2 + m^2}} = c(l, m).$$

Physically, this implies that the various waves of a composite signal will all travel at different speeds, causing a distortion of the signal over time. In particular, a localized burst of activity, which by virtue of the Fourier-decomposition theorem consists of waves of many different wavelengths, will be progressively less localized as time goes on. This phenomenon is called *dispersion*, and the mathematical function that relates the frequency ω to the wave-number components l and m bears the name of *dispersion relation*.

The dispersion relation can be represented, in two dimensions, as a set of curves in the (l, m) plane along which ω is a constant. Figure A-3 provides an example. At one dimension ($k = l$) or at two dimensions when the physical system is isotropic (ω

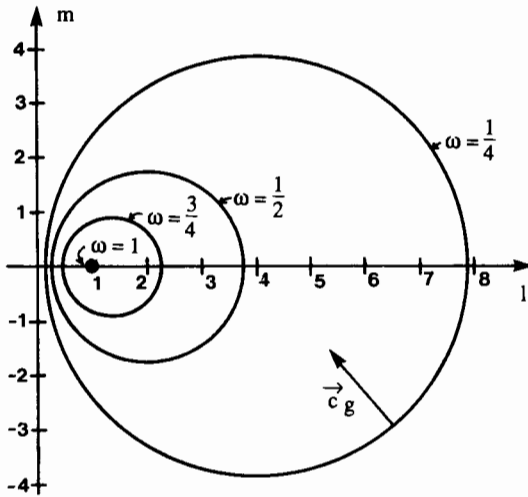


Figure A-3 Graphic representation of the dispersion relation $\omega = 2l/(l^2 + m^2 + 1)$ by curves of constant ω values in the (l, m) wave-number plane.

function of k only), a single ω -versus- k curve suffices to represent the dispersion relation.

In some special physical systems, the dispersion relation reduces to a single proportionality between frequency ω and wave number k . The phase speed is then the same for all wave numbers, all waves travel in perfect accord, and the total signal retains its shape as time evolves. Such a wave is called a *nondispersive wave*.

A-3 GROUP VELOCITY AND ENERGY PROPAGATION

In general, a wave pattern consists of more than a single wave. A series of waves are superimposed, leading to constructive and destructive interference. In areas where the waves are interfering constructively, the wave amplitude is greater and the energy level is higher than in areas where the waves interfere destructively. Therefore, energy distribution is a property of a set of waves rather than of a single wave. (A single wave has a uniform energy distribution.) Energy propagation by a set of waves depends on how interference patterns move about and is generally not the average speed of the waves present. To illustrate the principles and determine the speed at which energy propagates, let us restrict our attention to two one-dimensional waves and, more precisely, to two waves of equal amplitude and almost equal wave numbers:

$$a = A \cos(k_1 x - \omega_1 t) + A \cos(k_2 x - \omega_2 t), \tag{A-14}$$

where the wave numbers k_1 and k_2 are close to their average $k = (k_1 + k_2)/2$, and the difference $\Delta k = k_1 - k_2$ is much smaller ($|\Delta k| \ll |k|$). Because the two waves obey the same dispersion relation of the dynamical system, $\omega = \omega(k)$, the two frequencies $\omega_1 = \omega(k_1)$ and $\omega_2 = \omega(k_2)$ are close to their average $\omega = (\omega_1 + \omega_2)/2$, which is much larger than their difference $\Delta\omega = \omega_1 - \omega_2$ ($|\Delta\omega| \ll |\omega|$). In expression (A-14),

the two reference phases were set to zero, which can always be done under suitable choices of space and time origins.

A trigonometric manipulation transforms expression (A-14) into

$$a = 2A \cos\left(\frac{\Delta k}{2} x - \frac{\Delta \omega}{2} t\right) \cos(kx - \omega t), \quad (\text{A-15})$$

which now appears as the product of two waves. The second cosine function represents an average wave, of wave number and frequency between those of the two individual waves comprising the signal. The first cosine function, on the other hand, involves a much smaller wave number (i.e., much longer wavelength) and a much lower frequency. Over the cycle of the shorter (k, ω) wave, the longer wave appears almost unchanging. Thus, the (k, ω) wave appears modulated; its amplitude, $2A \cos[(\Delta kx - \Delta\omega t)/2]$, is slowly varying in space and time, as seen in Figure A-4.

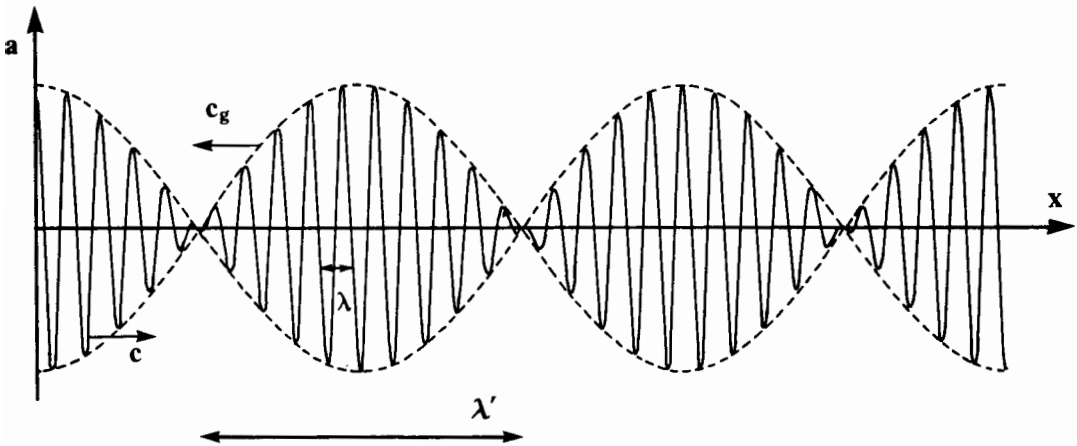


Figure A-4 The interference pattern of two one-dimensional waves with close wave numbers. While the wave crests and troughs propagate at the phase speed $c = \omega/k$, the envelope (dashed line) propagates at the group velocity $c_g = d\omega/dk$.

Although the wave signal exhibits a wavelength from crest to trough to the next crest equal to $\lambda = 2\pi/k$, the envelope has a much longer wavelength, $\lambda' = \frac{1}{2} [2\pi/(\Delta k/2)] = 2\pi/\Delta k$. The wave pattern is a succession of wave bursts, each of length λ' . Within each burst, the wave propagates at the phase speed $c = \omega/k$, while the burst travels at the speed $c' = \Delta\omega/\Delta k$.

Considering an infinitesimal wave-number difference, we are led to define

$$c_g = \frac{d\omega}{dk}. \quad (\text{A-16})$$

Because this is the propagation speed of a burst, or group of similar waves, it is called

the *group velocity*. Energy is associated with each group, and so the group velocity is also the velocity at which the energy is carried by the waves.

The preceding wave description relies on the existence of two waves of identical amplitudes. When two waves do not have equal amplitude, say A_1 and A_2 , destructive interference is nowhere complete (the weak wave cannot completely cancel the strong wave), and there is no clear pinch-off between wave bursts. Rather, the modulating envelope undulates between the values $A_1 + A_2$ and $|A_1 - A_2|$ on the positive side and $-(A_1 + A_2)$ and $-|A_1 - A_2|$ on the negative side. It remains, however, that regions of constructive interference, and thus of higher energy level, propagate at the group velocity.

The theory can easily be extended to multidimensional waves. For example, at two dimensions, we define the group velocities in the x - and y -directions, respectively, as

$$c_{gx} = \frac{\partial \omega}{\partial l}, \quad c_{gy} = \frac{\partial \omega}{\partial m}, \quad (\text{A-17})$$

given the dispersion relation $\omega(l, m)$. Because these expressions are the components of the gradient of the function ω in the (l, m) wave-number space, they can be interpreted as the components of a physical vector depicting the group velocity

$$\mathbf{c}_g = \nabla_k \omega, \quad (\text{A-18})$$

where ∇_k stands for the gradient operator with respect to the variables l and m . On the two-dimensional diagram (Figure A-3), this vector group velocity points perpendicularly to the ω curves, toward the higher values of ω . Aligning the l - and m -axes with the x - and y -axes of the plane provides the direction of energy propagation in space.

Generalization to the three-dimensional space is immediate. An example is the internal wave discussed extensively in Chapter 10.

PROBLEMS

- A-1.** In waters deeper than half the wavelength, surface gravity waves obey the dispersion relation $\omega = \sqrt{gk}$, where g is the gravitational acceleration ($g = 9.81 \text{ m/s}^2$). For these waves, show that the wavelength is proportional to the square of the period. At which speed does a 10-m-long wave travel?
- A-2.** Show that the group velocity of deep-water waves (see Problem A-2) is always less than the phase speed.
- A-3.** A former sea captain recounts a stormy night in the middle of the North Atlantic when he observed waves with wavelengths of a few meters passing his 51-m-long ship in less than 3 s. Should you believe him?
- A-4.** Suppose that you are in the middle of the ocean and off in the distance you see a storm. A little while later, you observe the passage of surface gravity waves of wavelength 5 m. Two hours later, you still observe gravity waves, but now their wavelength is 2 m. How far away was the storm?

- A-5.** Find the frequency ω of a Kelvin wave of wave number k (Section 6-2). Is the Kelvin wave dispersive?
- A-6.** Show that for inertia-gravity waves [$\omega^2 = f^2 + gH(l^2 + m^2)$; Section 6-3], the group velocity is always less than the phase speed. In which limit does the group velocity approach the phase speed?
- A-7.** Demonstrate that when the frequency ω is a function of the ratio l/m , the energy propagates perpendicularly to the phase.
- A-8.** Given the dispersion relation of internal waves in a vertical plane (Section 10-2),

$$\omega = N \frac{l}{\sqrt{l^2 + n^2}},$$

where N is a constant, l is the horizontal wave number and n is the vertical wave number, show that phase and energy always propagate in the same horizontal direction but in opposite vertical directions.

SUGGESTED COMPUTER DEMONSTRATION

Using animated graphics, display a time sequence ($t = 0$ to 10π by steps of $\pi/4$) of the double wave

$$a(x, t) = A_1 \cos(k_1 x - \omega_1 t) + A_2 \cos(k_2 x - \omega_2 t)$$

with $A_1 = A_2 = 1$, $k_1 = 1.9$, $k_2 = 2.1$, $\omega_1 = 2.1$, $\omega_2 = 1.9$, and for x ranging from 0 to 100. A suggested step in x is 0.5. Notice how the short waves [of wavelength = $4\pi/(k_1 + k_2) = \pi$] travel toward increasing x at the speed $c = (\omega_1 + \omega_2)/(k_1 + k_2) = +1$, while the wave envelope [of wavelength = $2\pi/(k_2 - k_1) = 10\pi$] travels in the opposite direction at speed $c_g = (\omega_1 - \omega_2)/(k_1 - k_2) = -1$. This unequivocally demonstrates the nonintuitive fact that the energy propagation may well propagate in the direction opposite to the advancing crests and troughs. In other words, it is not impossible for energy to be transported up-wave.

Variations can include uneven amplitudes (e.g., $A_1 = 1$ and $A_2 = 0.5$) and modified values for the wave numbers and frequencies.