15

Quasi-Geostrophic Dynamics

Summary: At time scales longer than about a day, geophysical flows are typically in a nearly geostrophic state, and it may be advantageous to capitalize on this property to obtain a simplified dynamical formalism. Here, we derive the traditional quasi-geostrophic dynamics and present some applications in both linear and nonlinear regimes.

15-1 SIMPLIFYING ASSUMPTION

Rotation effects become important when the Rossby number is on the order of unity or less (Sections 1-5 and 3-6). The smaller the Rossby number, the stronger the rotation effects and the larger the Coriolis force compared to the inertial force. In fact, the majority of atmospheric and oceanic motions are characterized by Rossby numbers sufficiently below unity ($Ro \sim 0.2$ down to 0.01) to enable us to state that, in first approximation, the Coriolis force is dominant. This leads to geostrophic equilibrium (Section 4-1), whereby a balance is struck between the Coriolis force and the pressure-gradient force. In Chapter 4 a theory was developed for perfectly geostrophic flows, whereas in Chapter 6 some near-geostrophic, small-amplitude waves were investigated.

In each case, the analysis was restricted to homogeneous flows. Here, we reconsider near-geostrophic motions but in the case of continuously stratified fluids and nonlinear dynamics. Much of the material presented can be traced to the seminal article by Charney (1948), which laid the foundation of quasi-geostrophic dynamics.

Geostrophic balance, which holds in first approximation, is a linear and diagnostic relationship (there is no product of variables and no time derivative). The resulting mathematical advantages explain why near-geostrophic dynamics are used routinely: The underlying assumption of near-geostrophy may not always be strictly valid, but the formalism is much simpler than otherwise.

Mathematically, a state of near-geostrophic balance occurs when the terms representing relative acceleration, nonlinear advection, and friction are all negligible in the horizontal momentum equations. This requires (Section 3-6) that the temporal Rossby number,

$$Ro_T = \frac{1}{\Omega T}, \tag{15-1}$$

the Rossby number,

$$Ro = \frac{U}{\Omega L}, \tag{15-2}$$

and the Ekman number,

$$Ek = \frac{\nu}{\Omega H^2}, \tag{15-3}$$

all be small simultaneously. In these expressions, Ω is the angular rotation rate of the earth (or planet or star under consideration), T is the time scale of the motion (i.e., the time span over which the flow field evolves substantially), U is a typical horizontal velocity in the flow, L is the horizontal length over which the flow extends or exhibits variations, ν is the vertical viscosity, and H is the height over which the flow extends.

The smallness of the Ekman number (Section 3-6) indicates that vertical friction is negligible, except perhaps in thin layers on the edges of the fluid domain (Chapter 5). If we exclude small-amplitude waves that can travel much faster than fluid particles in the flow, the temporal Rossby number (15-1) is not greater than the Rossby number (15-2). (For a discussion of this argument, the reader is referred to Section 6-1). By elimination, it remains to require that the Rossby number (15-2) be small. This can be justified in one of two ways: Either velocities are relatively weak (small U) or the flow pattern is laterally extensive (large L). The common approach, and the one that leads to the simplest formulation, is to consider the first possibility; the resulting formalism bears the name of *quasi-geostrophic dynamics*. We ought to keep in mind, however, that some atmospheric and oceanic motions could be nearly geostrophic for the other reason. Such motions would be improperly represented by quasi-geostrophic dynamics.

15-2 GOVERNING EQUATION

To set the stage for the development of quasi-geostrophic equations, it is most convenient to begin with the restriction that vertical displacements of density surfaces be small (Figure 15-1). In the (x, y, z) coordinate system, we write

$$\rho = \bar{\rho}(z) + \rho'(x, y, z, t), \qquad |\rho'| \ll |\bar{\rho}|. \tag{15-4}$$

In the (x, y, ρ) coordinate system, the equivalent statement is

$$z = \bar{z}(\rho) + z'(x, y, \rho, t), \quad |z'| \ll |\bar{z}|. \tag{15-5}$$

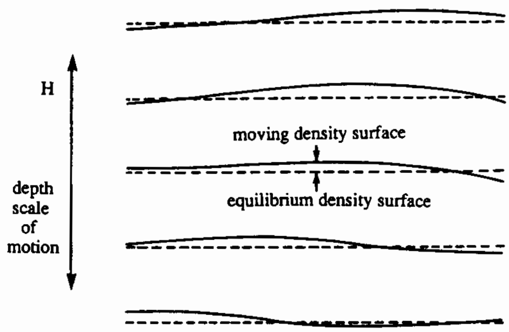

Figure 15-1 A stratified fluid experiencing weak motions, which can then be described by quasi-geostrophic dynamics.

Because the variables ρ and z are, in first approximation, uniquely related [via the function $\bar{\rho}(z)$ or its inverse $\bar{z}(\rho)$], there is no real advantage to be gained by using the density-coordinate system, and we follow the tradition here by formulating the quasi-geostrophic dynamics in the (x, y, z) Cartesian coordinate system.

The density profile $\bar{\rho}(z)$, independent of time and horizontally uniform, is called the basic stratification. Alone, it creates a state of rest in hydrostatic equilibrium. We shall assume that such stratification has somehow been established and that it is maintained in time against the homogenizing action of vertical diffusion. The quasi-geostrophic formalism does not consider the origin and maintenance of this stratification but only the behavior of motions that weakly perturb it.

The following mathematical developments are purposely heuristic, with emphasis on the exploitation of the main idea rather than on a systematic approach. The reader interested in a rigorous derivation of quasi-geostrophic dynamics based on a regular perturbation analysis is referred to Chapter 6 of the book by Pedlosky (1987).

The governing equations of Section 3-5 with $\rho = \bar{\rho}(z) + \rho'(x, y, z, t)$ and, similarly, $p = \bar{p}(z) + p'(x, y, z, t)$ are, on the beta plane,

$$\frac{du}{dt} - f_0 v - \beta_0 y v = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2}, \tag{15-6}$$

$$\frac{dv}{dt} + f_0 u + \beta_0 y u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2}, \tag{15-7}$$

$$0 = -\frac{\partial p'}{\partial z} - \rho' g, \tag{15-8}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{15-9}$$

$$\frac{\partial \rho'}{\partial t} + u\frac{\partial \rho'}{\partial x} + v\frac{\partial \rho'}{\partial y} + w\frac{d\bar{\rho}}{dz} = 0, \tag{15-10}$$

where the advective operator is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{15-11}$$

The basic assumption that $|\rho'|$ is much less than $|\bar{\rho}|$ has been implemented in the density equation (15-10) by dropping the term $w\partial\rho'/\partial z$. In writing that equation, we have also neglected vertical density diffusion (the right-hand side of the equation) in agreement with our premise that the basic vertical stratification persists.

If the density perturbations ρ' are small, so are the pressure disturbances p'; by virtue of the horizontal momentum equations, the horizontal velocities are weak. Although the Coriolis terms are small, the nonlinear advective terms, which involve products of velocities, are even smaller. For expediency, we shall use the phrase *very small* for these and all other terms smaller than the small terms. Thus, the ratio of advective to Coriolis terms, the Rossby number, is small. Let us assume now and verify a posteriori that the time scale is long compared to the inertial period $(2\pi/f_0)$, so the local-acceleration terms are, too, very small. Finally, to guarantee that the beta-plane approximation holds, we further require $|\beta_0 y| \ll f_0$. Having made all these assumptions, we take pleasure in noting that the dominant terms in the momentum equations are, as expected, those of the geostrophic equilibrium:

$$-f_0 v = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \tag{15-12a}$$

$$+f_0 u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}. \tag{15-12b}$$

As noted repeatedly in Chapter 4, this state is somewhat singular. In particular, it leads to a zero horizontal divergence ($\partial u/\partial x + \partial v/\partial y = 0$), which usually (e.g., over a flat bottom) implies the absence of any vertical velocity. In the case of a stratified fluid, this in turn implies no lifting and lowering of density surfaces and thus no pressure disturbances and no motion. The dynamics are degenerate, and the corrections brought by the neglected terms are essential. We replace u and v by their geostrophic values given by (15-12) in the very small terms of equations (15-6) and (15-7). The result is

$$-\frac{1}{\rho_0 f_0}\frac{\partial^2 p'}{\partial y \partial t} - \frac{1}{\rho_0^2 f_0^2} J\left(p', \frac{\partial p'}{\partial y}\right) - \frac{1}{\rho_0 f_0} w \frac{\partial^2 p'}{\partial y \partial z} - f_0 v$$
$$-\frac{\beta_0}{\rho_0 f_0} y \frac{\partial p'}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} - \frac{\nu}{\rho_0 f_0}\frac{\partial^3 p'}{\partial y \partial z^2}, \tag{15-13a}$$

$$+\frac{1}{\rho_0 f_0}\frac{\partial^2 p'}{\partial x \partial t}+\frac{1}{\rho_0^2 f_0^2} J\left(p', \frac{\partial p'}{\partial x}\right)+\frac{1}{\rho_0 f_0} w \frac{\partial^2 p'}{\partial x \partial z}+f_0 u$$

$$-\frac{\beta_0}{\rho_0 f_0} y \frac{\partial p'}{\partial y}=-\frac{1}{\rho_0}\frac{\partial p'}{\partial y}+\frac{\nu}{\rho_0 f_0}\frac{\partial^3 p'}{\partial x \partial z^2}. \tag{15-13b}$$

The symbol $J(\,,\,)$ represents the Jacobian operator, which is defined as $J(a, b) = \partial a/\partial x\ \partial b/\partial y - \partial a/\partial y\ \partial b/\partial x$. From these equations, more accurate expressions for u and v can readily be extracted. The improved flow field has a nonzero divergence, which is small because it is due only to the weak velocity departures from the otherwise nondivergent geostrophic flow. Thus, the vertical velocity is nonzero but very small, and the term of the advection operator (15-11) containing the vertical velocity brings a correction to corrections. Dropping the corresponding w-terms from equations (15-13) and solving for u and v, we obtain

$$u=-\frac{1}{\rho_0 f_0}\frac{\partial p'}{\partial y}-\frac{1}{\rho_0 f_0^2}\frac{\partial^2 p'}{\partial x \partial t}-\frac{1}{\rho_0^2 f_0^3} J\left(p', \frac{\partial p'}{\partial x}\right)$$

$$+\frac{\beta_0}{\rho_0 f_0^2} y \frac{\partial p'}{\partial y}+\frac{\nu}{\rho_0 f_0^2}\frac{\partial^3 p'}{\partial x \partial z^2}, \tag{15-14a}$$

$$v=+\frac{1}{\rho_0 f_0}\frac{\partial p'}{\partial x}-\frac{1}{\rho_0 f_0^2}\frac{\partial^2 p'}{\partial y \partial t}-\frac{1}{\rho_0^2 f_0^3} J\left(p', \frac{\partial p'}{\partial y}\right)$$

$$-\frac{\beta_0}{\rho_0 f_0^2} y \frac{\partial p'}{\partial x}+\frac{\nu}{\rho_0 f_0^2}\frac{\partial^3 p'}{\partial y \partial z^2}, \tag{15-14b}$$

which, unlike (15-12a) and (15-12b), contain both the geostrophic flow and a series of first ageostrophic corrections. Upon substitution of these expressions in continuity equation (15-9), we obtain

$$\frac{\partial w}{\partial z}=\frac{1}{\rho_0 f_0^2}\left[\frac{\partial}{\partial t}\nabla^2 p'+\frac{1}{\rho_0 f_0} J(p', \nabla^2 p')+\beta_0 \frac{\partial p'}{\partial x}-\nu \nabla^2 \frac{\partial^2 p'}{\partial z^2}\right], \tag{15-15}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the two-dimensional Laplacian operator.

We now turn our attention to the density-conversation equation (15-10). The first term is very small not only because ρ' is small but also because the time scale is long. Likewise, the last term is very small because, as we concluded before, the vertical velocity arises from the ageostrophic corrections to the already weak horizontal velocity. The middle terms involve the density perturbation, which is small, and the horizontal velocities, which are also small. There is thus no need, in this equation, for the corrections brought by (15-14) and the geostrophic expressions (15-12) suffice, leaving

$$\frac{\partial \rho'}{\partial t}+\frac{1}{\rho_0 f_0} J(p', \rho')-\frac{\rho_0 N^2}{g} w=0, \tag{15-16}$$

where the stratification frequency, $N^2(z) = -\ (g/\rho_0)\ d\bar{\rho}/dz$, has been introduced.

Dividing this last equation by N^2/g, taking its z-derivative, and using hydrostatic equation (15-8) to eliminate density, we obtain

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right] + \frac{1}{\rho_0 f_0} J\left[p', \frac{\partial}{\partial z}\left(\frac{1}{N^2}\frac{\partial p'}{\partial z}\right)\right] + \rho_0 \frac{\partial w}{\partial z} = 0. \quad (15\text{-}17)$$

Equations (15-15) and (15-17) form a two-by-two system for the perturbation pressure p' and vertical stretching $\partial w/\partial z$. Elimination of $\partial w/\partial z$ yields a final, single equation for p':

$$\frac{\partial}{\partial t}\left[\nabla^2 p' + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial p'}{\partial z}\right)\right] + \frac{1}{\rho_0 f_0} J\left[p', \nabla^2 p' + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial p'}{\partial z}\right)\right] + \beta_0 \frac{\partial p'}{\partial x} = \nu \frac{\partial^2}{\partial z^2}\nabla^2 p'. \quad (15\text{-}18)$$

This is the quasi-geostrophic equation for nonlinear motions in a continuously stratified fluid on a beta plane. Usually, this equation is recast as an equation for the potential vorticity, and the pressure field is transformed into a streamfunction via $p' = \rho_0 f_0 \psi$. The result is

$$\frac{\partial q}{\partial t} + J(\psi, q) = \nu \frac{\partial^2 \nabla^2 \psi}{\partial z^2}, \quad (15\text{-}19)$$

where q is the potential vorticity:

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) + \beta_0 y. \quad (15\text{-}20)$$

Once the solution is obtained, the original variables are derived from (15-8), (15-12), and (15-16):

$$u = -\frac{\partial\psi}{\partial y} \quad (15\text{-}21a)$$

$$v = +\frac{\partial\psi}{\partial x} \quad (15\text{-}21b)$$

$$w = -\frac{f_0}{N^2}\left[\frac{\partial^2\psi}{\partial t\,\partial z} + J\left(\psi, \frac{\partial\psi}{\partial z}\right)\right] \quad (15\text{-}21c)$$

$$p' = \rho_0 f_0\,\psi \quad (15\text{-}21d)$$

$$\rho' = -\frac{\rho_0 f_0}{g}\frac{\partial\psi}{\partial z}. \quad (15\text{-}21e)$$

15-3 DISCUSSION

Expression (15-20) indicates that q is a potential-vorticity expression. Indeed, the last term represents the planetary contribution to the vorticity, whereas the first term, $\nabla^2\psi = \partial v/\partial x - \partial u/\partial y$, is the relative vorticity. The middle term can be traced to the layer-thickness variations in the denominator of the classical definition of potential vorticity [e.g. (12-18)]. It is thus the contribution of vertical stretching in some linearized form.

It is most interesting to compare the first two terms of the potential-vorticity expression, namely, relative vorticity and vertical stretching. With L and U as the horizontal length and velocity scales, respectively, the streamfunction ψ scales like LU by virtue of (15-21a) and (15-21b). If H is the vertical length scale, the magnitudes of those contributions to potential vorticity are

$$\text{Relative vorticity} \sim \frac{U}{L}, \quad \text{Vertical stretching} \sim \frac{f_0^2 UL}{N^2H^2}.$$

The ratio of the former to the latter is

$$\frac{\text{Relative vorticity}}{\text{Vertical stretching}} \sim \frac{N^2H^2}{f_0^2L^2} = Bu,$$

which is the Burger number defined in Section 9-5. For small Burger numbers ($NH \ll f_0L$), that is, weak stratification or long length scales, vertical stretching dominates, and the motion is akin to that of homogeneous rotating flows in nearly geostrophic balance (Chapter 4), where topographic variations are capable of exerting great influence. For large Burger numbers ($NH \gg f_0L$), that is, strong stratification or short length scales, relative vorticity dominates, stratification reduces coupling in the vertical, and each level tends to behave in a two-dimensional fashion, stirred by its own vorticity pattern.

The richest behavior occurs when the stratification and length scale match to make the Burger number of order unity, which occurs when

$$L = \frac{NH}{f_0}. \tag{15-22}$$

As noted in Section 12-3, this particular length scale is the internal radius of deformation. To show this, let us introduce a nominal density difference $\Delta\rho$, typical of the density vertical variations of the ambient stratification. Thus, $|d\bar{\rho}/dz| \sim \Delta\rho/H$ and $N^2 \sim g\Delta\rho/\rho_0 H$. Defining the reduced gravity $g' = g\Delta\rho/\rho_0$, which is typically much less than the full gravity, we obtain

$$N \sim \sqrt{\frac{g'}{H}}. \tag{15-23}$$

Definition (15-22) then yields

$$L \sim \frac{\sqrt{g'H}}{f_0}. \tag{15-24}$$

Comparing this expression with definition (6-10) for the radius of deformation in homogeneous rotating fluids, we note the replacement of the full gravitational acceleration by a much smaller, reduced acceleration and conclude that motions in stratified fluids tend to take place on shorter scales than dynamically similar motions in homogeneous fluids.

Before concluding this section, it is noteworthy to return to the discussion of the time scale. Very early in the derivation, an assumption was made to consider only slowly evolving motions, namely, motions with time scale T much longer than the inertial time scale $1/f_0$ ($T \gg \Omega^{-1}$). This relegated the terms $\partial u/\partial t$ and $\partial v/\partial t$ to the rank of small perturbations to the dominant geostrophic balance. Now, having completed our analysis, we ought to check for consistency.

The time scale of quasi-geostrophic motions can be most easily determined by inspection of the governing equation in its potential-vorticity form. Barring an unlikely dominance of dissipation, the balance of (15-19) requires that the two terms on its left-hand side be on the same order:

$$\frac{Q}{T} \sim \frac{LU}{L}\frac{Q}{L},$$

where Q is the scale of potential vorticity, regardless of whether it is dominated by relative vorticity ($Q \sim U/L$) or vertical stretching ($Q \sim f_0^2 UL/N^2H^2$), and LU is the streamfunction scale. The preceding statement yields

$$T \sim \frac{L}{U}; \tag{15-25}$$

in other words, the time scale is advective. The quasi-geostrophic structure evolves on a time T comparable to the time taken by a particle to cover the length scale L at the nominal speed U. For example, a vortex flow (such as an atmospheric cyclone) evolves significantly while particles complete one revolution.

Because the quasi-geostrophic formalism is rooted in the smallness of the Rossby number ($Ro = U/\Omega L \ll 1$), it follows directly that the time scale must be long compared with the rotation period:

$$T \gg \frac{1}{\Omega}, \tag{15-26}$$

in agreement with our premise. Note, however, that a lack of contradiction is a proof only of consistency in the formalism. It implies that slowly evolving, quasi-geostrophic motions can exist, but the existence of other, non-quasi-geostrophic motions are certainly not precluded. Among the latter, we can distinguish nearly geostrophic motions of other types (Phillips, 1963; Cushman-Roisin, 1986; Cushman-Roisin et al., 1992) and, of course, completely ageostrophic motions (see the examples in Chapters 10 and 14).

Whereas ageostrophic flows typically evolve on the inertial time scale ($T \sim \Omega^{-1}$), geostrophic motions of type other than quasi-geostrophy usually evolve on much longer time scales ($T \gg L/U \gg \Omega^{-1}$).

15-4 ENERGETICS

Because the quasi-geostrophic formalism is frequently used, it is worth investigating the approximate energy budget to which it corresponds. Multiplying the governing equation (15-19) by the streamfunction ψ and integrating over the entire three-dimensional domain, we obtain, after several integrations by parts:

$$\frac{d}{dt}\left[\iiint \frac{1}{2}\,\rho_0 |\nabla\psi|^2\, dx\, dy\, dz + \iiint \frac{1}{2}\,\rho_0\, \frac{f_0^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2 dx\, dy\, dz\right]$$
$$= -\rho_0 \nu \iiint \left|\nabla \frac{\partial\psi}{\partial z}\right|^2 dx\, dy\, dz. \qquad (15\text{-}27)$$

The boundary terms have all been set to zero by assuming rigid bottom and top surfaces, vertical meridional walls, and decay at large distances (or periodicity) in the zonal direction. Equation (15-27) can be interpreted as an energy statement: The sum of kinetic and potential energies (the two integrals following the time derivative) decays in time because of mechanical dissipation (term on the right, always negative). That the first integral corresponds to kinetic energy is evident once the velocity components have been expressed in terms of the streamfunction ($u^2 + v^2 = \psi_y^2 + \psi_x^2 = |\nabla\psi|^2$). By default, this leaves the second integral to play the role of potential energy, which is not evident. Basic physical principles would indeed suggest the following definition for potential energy:

$$PE = \iiint \rho g z\, dx\, dy\, dz,$$

which by virtue of (15-21e) would yield an expression linear, not quadratic, in ψ.

The discrepancy is resolved by defining the *available potential energy*, a concept first advanced by M. Margules (1903) and developed by E. N. Lorenz (1955). Because the fluid occupies a fixed volume, the rising of fluid in some locations must be accompanied by a descent of fluid elsewhere; therefore, any potential-energy gain somewhere is necessarily compensated, at least partially, by a potential-energy drop elsewhere. What matters then is not the total potential energy of the fluid but only how much could be converted from the instantaneous, perturbed density distribution. We define the available potential energy, *APE*, as the difference between the existing potential energy, as just defined, and the potential energy that the fluid would have if the basic stratification were unperturbed.

The situation is best illustrated in the case of a two-layer stratification (Figure 15-2): A lighter fluid of density ρ_1 floats atop of a denser fluid of density ρ_2. In the presence of motions, the interface is at level h above the resting height H_2 of the lower

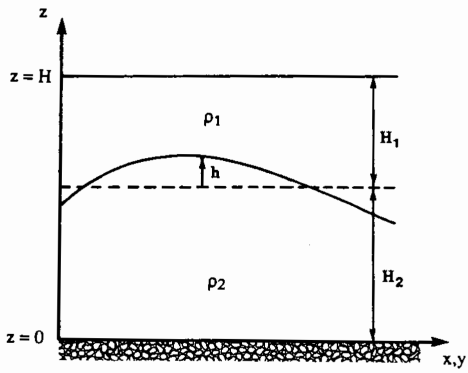

Figure 15-2 A two-layer stratification, for the illustration of the concept of available potential energy.

layer. Because of volume conservation, the integral of h over the horizontal domain vanishes identically. The potential energy associated with the perturbed state is

$$PE(h)$$

$$= \int\int dx\ dy \left[\int_0^{H_2+h} \rho_2 g z\ dz + \int_{H_2+h}^{H} \rho_1 g z\ dz\right]$$

$$= \int\int \left[\frac{1}{2}\ \rho_1 g H^2 + \frac{1}{2}\ \Delta\rho g H_2^2\right] dx\ dy + \int\int \Delta\rho H_2 h\ dx\ dy + \int\int \frac{1}{2}\ \Delta\rho g h^2\ dx\ dy,$$

where H is the total height and $\Delta\rho = \rho_2 - \rho_1$ is the density difference. The first term represents the potential energy in the unperturbed state, whereas the second term vanishes because h has a zero mean. It leaves the third term as the available potential energy:

$$APE = PE(h) - PE(h = 0)$$

$$= \int\int \frac{1}{2}\ \Delta\rho g h^2\ dx\ dy. \qquad (15\text{-}28)$$

Introducing the stratification frequency $N^2 = -(g/\rho_0)\ d\bar{\rho}/dz = g\Delta\rho/\rho_0 H$ and generalizing to three dimensions, we obtain

$$APE = \int\int\int \frac{1}{2}\ \rho_0 N^2 h^2\ dx\ dy\ dz. \qquad (15\text{-}29)$$

In a continuous stratification, the vertical displacement h of a fluid parcel is directly related to the density perturbation—that is, the difference between the density of the parcel that comes from a distance h, as given here, and the density that would otherwise exist there:

$$\rho'(x, y, z, t) = \bar{\rho}[z - h(x, y, z, t)] - \bar{\rho}(z)$$

$$\simeq -h\ \frac{d\bar{\rho}}{dz} = \frac{\rho_0 N^2}{g}\ h. \qquad (15\text{-}30)$$

This use of a Taylor approximation is justified by the underlying assumption of weak vertical displacements. Combining (15-29) and (15-30) and expressing the density perturbation in terms of the streamfunction by (15-21e), we obtain

$$APE = \iiint \frac{1}{2} \rho_0 \frac{f_0^2}{N^2} \left(\frac{\partial \psi}{\partial z}\right)^2 dx\, dy\, dz.$$

which is the integral that arises in the energy budget, (15-27).

As a final note, we observe that the time rate of change of the available potential energy can be expressed as

$$\frac{d}{dt} APE = g \iiint \rho' w \; dx\, dy\, dz,$$

which shows that the potential energy increases when, on the average, heavy fluid parcels rise (ρ' and w both positive) and light parcels sink (ρ' and w both negative).

15-5 PLANETARY WAVES IN A STRATIFIED FLUID

In Chapter 6, it was noted that inertia-gravity waves are superinertial ($\omega \geq f$) and that Kelvin waves require a fundamentally ageostrophic balance in one of the two horizontal directions [see equation (6-5) with $u = 0$]. Therefore, the quasi-geostrophic formalism cannot describe these two types of waves. It can, however, very well describe the slow waves and, in particular, the planetary waves that exist on the beta plane.

It is instructive to explore the three-dimensional behavior of planetary (Rossby) waves in a continuously stratified fluid. The theory proceeds from the linearization of the quasi-geostrophic equation and, for mathematical simplicity only, the assumptions of a constant stratification frequency and negligible dissipation. Equations (15-19) and (15-20) then yield

$$\frac{\partial}{\partial t}\left(\nabla^2 \psi + \frac{f_0^2}{N^2}\frac{\partial^2 \psi}{\partial z^2}\right) + \beta_0 \frac{\partial \psi}{\partial x} = 0. \qquad (15\text{-}31)$$

We seek a wave solution of the form $\psi(x, y, z, t) = a(z)\cos(lx + my - \omega t)$, of horizontal wave numbers l and m, frequency ω, and vertical amplitude profile $a(z)$. This profile is governed by

$$\frac{d^2 a}{dz^2} - \frac{N^2}{f_0^2}\left(l^2 + m^2 + \frac{\beta_0 l}{\omega}\right) a = 0, \qquad (15\text{-}32)$$

which results from the substitution of the wave solution into (15-31). To solve this equation, boundary conditions are necessary in the vertical. For simplicity, let us assume that our fluid is bounded below by a horizontal surface and above by a free surface. n the atmosphere, this situation would correspond to the troposphere above a flat terrain or sea and below the tropause. At the bottom (say, $z = 0$), the vertical velocity vanishes, and the linearized form of (15-21c) implies $\partial^2 \psi / \partial z \partial t = 0$, or

$$\frac{da}{dz} = 0 \quad \text{at} \quad z = 0. \tag{15-33}$$

At the free surface (say, $z = h$), the pressure is to be uniform. Because the total pressure consists of the hydrostatic pressures due to the reference density ρ_0 (eliminated when the Boussinesq approximation was made; see Section 3-3) and to the basic stratification $\bar{\rho}(z)$, together with the perturbation pressure caused by the wave, we write:

$$P_0 - \rho_0 g z + g \int_z^h \bar{\rho}(z')\, dz' + p'(x, y, h, t) = \text{constant},$$

at the free surface $z = h(x, y, t)$. Because particles on the free surface remain on the free surface at all times (there is no inflow/outflow), we also state

$$w = \frac{\partial h}{\partial t} + u\, \frac{\partial h}{\partial x} + v\, \frac{\partial h}{\partial y},$$

at $z = h$. The preceding two statements are then linearized. Writing $h = H + \eta$, where the free-surface displacement $\eta(x, y, t)$ is small to justify linear wave motions, we expand the variables p' and w in the Taylor fashion from the mean surface level $z = H$ and systematically drop all terms involving products of variables of the wave field. The two requirements then reduce to

$$-\rho_0 g \eta + p' = 0 \quad \text{and} \quad w = \frac{\partial \eta}{\partial t} \quad \text{at } z = H. \tag{15-34}$$

Elimination of η yields $\partial p'/\partial t = \rho_0 g w$; in terms of the streamfunction,

$$\frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial z} + \frac{N^2}{g}\, \psi\right) = 0 \quad \text{at } z = H \tag{15-35}$$

or, finally, in terms of the wave amplitude,

$$\frac{da}{dz} + \frac{N^2}{g}\, a = 0 \quad \text{at } z = H. \tag{15-36}$$

Together, equation (15-32) and its two boundary conditions, (15-33) and (15-36), define an eigenvalue problem, which admits solutions of the form

$$a(z) = A \cos nz. \tag{15-37}$$

Substitution of this solution into equation (15-32) yields the dispersion relation linking the wave frequency ω to the horizontal and vertical wave numbers, l, m, and n:

$$\omega = -\frac{\beta_0 l}{l^2 + m^2 + n^2 f_0^2 / N^2}, \tag{15-38}$$

whereas substitution in boundary condition (15-36) imposes a condition on the wave number n:

$$\tan nH = \frac{N^2H}{g}\,\frac{1}{nH}. \qquad (15\text{-}39)$$

As Figure 15-3 demonstrates graphically, there is an infinite number of discrete solutions. Because negative values of n lead to solutions identical to those with positive n values [see (15-37) and (15-38)], it is necessary to consider only the latter set of values ($n > 0$).

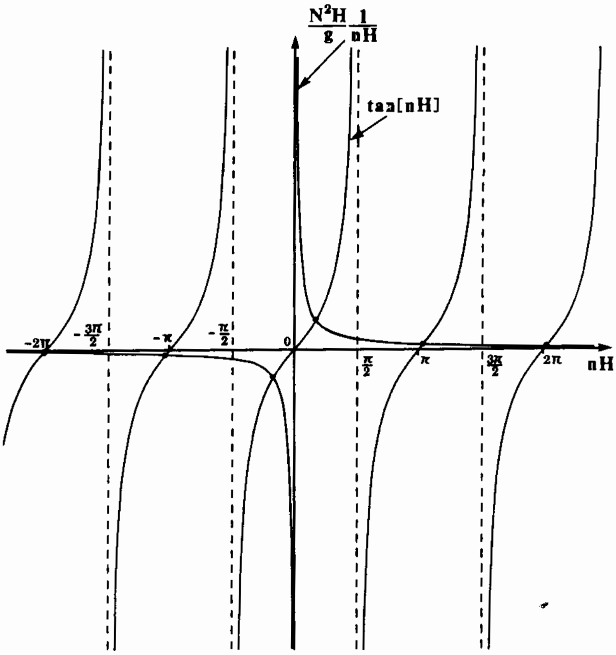

Figure 15-3 Graphical solution of equation (15-35). Every crossing of curves yields an acceptable value for the vertical wave number n. The pair of values nearest to the origin corresponds to a solution fundamentally different from all others.

A return to the definition $N^2 = -\,(g/\rho_0)\; d\bar{\rho}/dz$ reveals that the ratio N^2H/g, appearing on the right-hand side of (15-39), is equal to $\Delta\rho/\rho_0$, where $\Delta\rho$ is the density difference between top and bottom of the basic stratification $\bar{\rho}(z)$. The factor N^2H/g is thus very small, implying that the first solution of (15-39) falls very near the origin (Figure 15-3). There, $\tan nH$ can be approximated to nH, yielding

$$n = \frac{N}{\sqrt{gH}}.$$

Because nH is small, the corresponding wave is nearly uniform in the vertical. Its dispersion relation, obtained from the substitution of the preceding value of n into (15-38),

$$\omega = -\,\frac{\beta_0 l}{l^2 + m^2 + f_0^2/gH}, \qquad (15\text{-}40)$$

is independent of the stratification frequency N and identical to the dispersion relation obtained for planetary waves in homogeneous fluids [see (6-22)]. From this, we conclude that this wave is the barotropic component of the wave set.

The other solutions for n can also be determined to the same degree of approximation. Because N^2H/g is small, the finite solutions of (15-39) fall very near the zeros of tan nH (Figure 15-3) and are thus given approximately by

$$n_i = i\,\frac{\pi}{H}, \qquad i = 1, 2, 3, \ldots .$$

Unlike the first wave, the waves with these wave numbers exhibit substantial variations in the vertical and can be called baroclinic. Their dispersion relation,

$$\omega_i = -\frac{\beta_0 l}{l^2 + m^2 + (i\pi f_0/NH)^2}, \qquad (15\text{-}41)$$

is morphologically identical to (15-40), implying that they, too, are planetary waves. In summary, the presence of stratification permits the existence of an infinite, discrete set of planetary waves, one barotropic and all other baroclinic.

Comparing the dispersion relations (15-40) and (15-41) of the barotropic and baroclinic waves, we note the replacement in the denominator of the ratio f_0^2/gH by a multiple of $(\pi f_0/NH)^2$, which is much larger, since—again—N^2H/g is very small. Physically, the barotropic component is influenced by the large, external radius of deformation $\sqrt{gH}/f_0$ [see (6-10)], whereas the baroclinic waves feel the shorter, internal radius of deformation NH/f_0 [see (15-22)].

In the atmosphere, there is not always a great disparity between the two radii of deformation. Take, for example, a midlatitude region (such as 45°N, where $f_0 = 1.03 \times 10^{-4}\ \text{s}^{-1}$), a tropospheric height $H = 10$ km, and a stratification frequency $N = 0.01\ \text{s}^{-1}$. This yields $\sqrt{gH}/f_0 = 3050$ km and $NH/f_0 = 972$ km. (The ratio N^2H/g is then 0.102, which is not very small.) In contrast, the difference between the two radii of deformation is much more pronounced in the ocean. Take, for example, $H = 3$ km and $N = 2 \times 10^{-3}\ \text{s}^{-1}$, which yield $\sqrt{gH}/f_0 = 1670$ km and $NH/f_0 = 58$ km.

In any event, all planetary waves exhibit a zonal phase speed. For the baroclinic members of the family, it is

$$c_i = \frac{\omega_i}{l} = -\frac{\beta_0}{l^2 + m^2 + (i\pi f_0/NH)^2}. \qquad (15\text{-}42a)$$

Because this quantity is always negative, the direction can only be westward. (The meridional phase speed, ω_i/m, may be either positive or negative, depending on the sign of m.) Moreover, the westward speed is confined to the interval

$$-\beta_0 R_i^2 < c_i < 0, \qquad (15\text{-}42b)$$

with the lower bound approached by the longest wave ($l^2 + m^2 \to 0$). The quantities R_i, defined as

$$R_i = \frac{1}{i} \frac{NH}{\pi f_0}, \quad i = 1, 2, 3, \ldots, \tag{15-43}$$

are identified as internal radii of deformation, one for each baroclinic mode. The greater the value of i, the greater the value of n_i, the more reversals the wave exhibits in the vertical, and the more restricted is its zonal propagation. Therefore, the waves most active in transmitting information and carrying energy from east to west (or from west to east, if the group velocity is positive) are the barotropic and the first baroclinic components. Indeed, observations reveal that these two modes alone carry generally 80% to 90% of the energy in the ocean.

Let us now turn our attention to the spatial structure of a baroclinic planetary wave. For simplicity, we take the first mode ($i = 1$), which corresponds to a wave with one reversal of the flow in the vertical, and we set m to zero to focus on the zonal profile of the wave. The streamfunction, velocity, pressure, and density distributions are as follows:

$$\psi = A \cos nz \cos(lx - \omega t) \tag{15-44a}$$

$$u = -\frac{\partial \psi}{\partial y} = 0 \tag{15-44b}$$

$$v = +\frac{\partial \psi}{\partial x} = -lA \cos nz \sin(lx - \omega t) \tag{15-44c}$$

$$w = -\frac{f_0}{N^2} \frac{\partial^2 \psi}{\partial t \partial z} = +\frac{f_0 \omega n}{N^2} A \sin nz \sin(lx - \omega t) \tag{15-44d}$$

$$p' = \rho_0 f_0 \psi = \rho_0 f_0 A \cos nz \cos(lx - \omega t) \tag{15-44e}$$

$$\rho' = -\frac{\rho_0 f_0}{g} \frac{\partial \psi}{\partial z} = +\frac{\rho_0 f_0 n}{g} A \sin nz \cos(lx - \omega t). \tag{15-44f}$$

Because the geostrophic zonal velocity, u, is identically zero, it is necessary to consider its leading ageostrophic component. According to (15-14a) and our subsequent restrictions to linear and inviscid dynamics, we find this component to be

$$u = -\frac{1}{\rho_0 f_0^2} \frac{\partial^2 p'}{\partial t \partial x}$$

$$= -\frac{l\omega}{f_0} A \cos nz \cos(lx - \omega t). \tag{15-45}$$

The corresponding wave structure is displayed in Figure 15-4 and can be interpreted as follows.

At the top and bottom, vertical displacements are prohibited, and there is no density anomaly. In between, however, vertical displacements do occur and, for the gravest baroclinic mode, they exhibit one maximum at midlevel. Where the middle density surface upwells, heavier (colder) fluid from below is brought upward, forming a cold

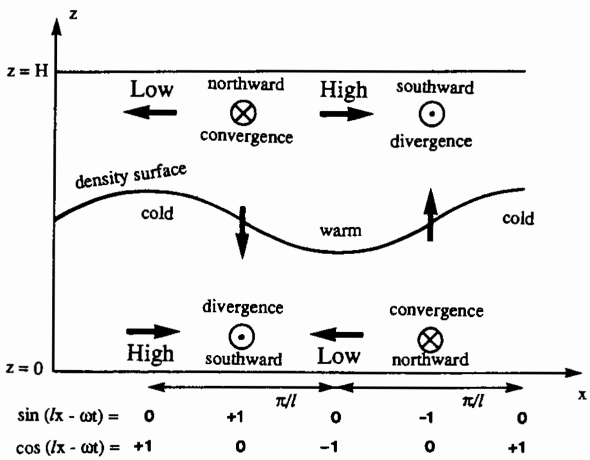

Figure 15-4 Structure of a baroclinic planetary wave. In constructing this diagram, we have taken $f_0 > 0$, $l > 0$, $m = 0$, and $n = \pi/H$, which yield $\omega < 0$ and a wave structure with one reversal in the vertical and velocities in the directions depicted above.

anomaly. Similarly, a warm anomaly accompanies a subsidence, half a wavelength away. Because colder fluid is heavier and warmer fluid is lighter, the bottom pressure is higher under cold anomalies and lower under warm anomalies. The resulting zonal pressure gradient drives an alternating meridional flow. In the Northern Hemisphere (as depicted in Figure 15-4), the velocity has the higher pressure on its right and therefore assumes a southward direction east of the high pressures and a northward direction east of the low pressures. Due to the baroclinic nature of the wave, there is a reversal in the vertical, and the velocities near the top are counter to those below (Figure 15-4).

On the beta plane, the variation in the Coriolis parameter causes this meridional flow to be convergent or divergent. In the Northern Hemisphere, the northward increase of f implies, under a uniform pressure gradient, a decreasing velocity and thus convergence of northward flow and divergence of southward flow. The resulting convergence-divergence pattern calls for transverse velocities, either zonal or vertical or both. According to Figure 15-4, based on (15-44d) and (15-45), both transverse components come into play, each partially relieving the convergence-divergence of the meridional flow. The ensuing vertical velocities at midlevel cause subsidence below a convergence and above a divergence, feeding the excess of the upper flow into the deficit of that underneath, and create upwelling half a wavelength away, where the situation is vertically reverse. Subsidence generates a warm anomaly, and upwelling generates a cold anomaly. As we can see in Figure 15-4, this takes place a quarter of a wavelength to the west of the existing anomalies, thus inducing a westward shift of the wave pattern over time. The result is a wave pattern steadily translating to the west.

15-6 SOME NONLINEAR EFFECTS

In its original form, the quasi-geostrophic equation (15-19) is quadratic in the streamfunction. An assumption of weak amplitudes was therefore necessary to explore the linear wave regime, and it is proper to ask now what role nonlinearities could play. For evident reasons, no general solution of the nonlinear equation is available. Nonlinearities cause interactions among the existing waves, generating harmonics and spreading the energy over a wide spectrum of scales. According to numerical simulations (Rhines, 1977; McWilliams, 1989), the result is a complicated unsteady state of motion, which has been termed *geostrophic turbulence*.

Although this topic will be better expounded in a later chapter (Section 17-3), it is worth mentioning here the natural tendency of geostrophic turbulence to form coherent structures (McWilliams, 1984; 1989). These take the form of distinct and robust vortices, which can be clearly identified and traced for periods of time long compared to their turnaround times. Figure 15-5 provides an example. The vortices contain a disproportionate amount of the energy available, being, therefore, highly nonlinear and leaving a relatively weak and linear wavefield in the intermediate space. In other words,

Figure 15-5 Vorticity distribution at some time during the evolution of geostrophic turbulence according to the quasi-geostrophic equation. Note the existence of identifiable vortices. (From McWilliams, 1989.)

a mature state of geostrophic turbulence displays a dichotomous pattern of nonlinear, localized vortices and linear, nonlocalized waves.

To explore nonlinear effects, let us seek a localized, vortex-type solution of finite amplitude. To simplify the analysis, we make the following assumptions of inviscid fluid ($\nu = 0$) and uniform stratification ($N =$ constant). Furthermore, expecting a possible zonal drift reminiscent of planetary waves, we seek solutions that are steadily translating in the x-direction. Thus, we state

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \tag{15-46}$$

$$q = \nabla^2 \psi + \frac{f_0^2}{N^2} \frac{\partial^2 \psi}{\partial z^2} + \beta_0 y, \tag{15-47}$$

where $\psi = \psi(x - ct, y, z)$ is a function vanishing at large distances.

Because the variables x and t occur only in the combination $x - ct$, the time derivative can be assimilated to an x-derivative ($\partial / \partial t = - c \partial / \partial x$) and the equation becomes

$$J(\psi + cy, q) = 0,$$

which admits the general solution

$$q = \nabla^2 \psi + \frac{f_0^2}{N^2} \frac{\partial^2 \psi}{\partial z^2} + \beta_0 y = F(\psi + cy). \tag{15-48}$$

The function F is, at this stage, an arbitrary function of its variable, $\psi + cy$. Because the vortex is required to be localized, the streamfunction must vanish at large distances, including large zonal distances at finite values of the meridional coordinate y. From (15-48), this implies

$$\beta_0 y = F(cy),$$

and the function F is linear: $F(\alpha) = (\beta_0 / c)\alpha$. Naturally, the function F may be multivalued, taking values along contours of $\psi + cy$ not connected to infinity (and, therefore, closed onto themselves within the confines of the vortex) that are different from the values along other, open contours of $\psi + cy$. In other words, the same $\psi + cy$ on two different contours could correspond to two distinct values of the function F.

Mindful of this possibility but restricting our attention for now to the region extending to infinity, where F is linear, we have, from (15-44),

$$\nabla^2 \psi + \frac{f_0^2}{N^2} \frac{\partial^2 \psi}{\partial z^2} = \frac{\beta_0}{c} \psi. \tag{15-49}$$

Now, assuming the existence of rigid surfaces at the top and bottom, we impose $\partial \psi / \partial z = 0$ at $z = 0, H$; only certain vertical modes are allowed. The gravest of these has the structure $\psi = a(r, \theta) \cos(\pi z / H)$, where (r, θ) are the polar coordinates associated with the Cartesian coordinates $(x - ct, y)$. The horizontal structure of the solution is prescribed by the amplitude $a(r, \theta)$, which must satisfy

$$\frac{\partial^2 a}{\partial r^2} + \frac{1}{r}\frac{\partial a}{\partial r} - \frac{1}{r^2}\frac{\partial^2 a}{\partial \theta^2} - \left(\frac{1}{R^2} + \frac{\beta_0}{c}\right) a = 0, \tag{15-50}$$

by virtue of (15-49), where

$$R = \frac{NH}{\pi f_0}, \tag{15-51}$$

is the internal radius of deformation. (The factor π is introduced here for convenience.) Such as equation admits solutions consisting of sinusoidal functions in the azimuthal direction and Bessel functions in the radial direction.

Because the potential energy is proportional to the integrated square of the vertical displacements, a localized vortex structure of finite energy requires a streamfunction field (proportional to a) that decays at large distances faster than $1/r$. This requirement excludes the Bessel functions of the first kind, which decay only as $1/r^{1/2}$, and leaves us with the modified Bessel functions, which decay exponentially:

$$a(r, \theta) = \sum_{m=0}^{\infty} (A_m \cos m\theta + B_m \sin m\theta)\, K_m(kr), \tag{15-52}$$

where the factor k, defined by

$$k^2 = \frac{1}{R^2} + \frac{\beta_0}{c} \tag{15-53}$$

must be real. The condition that k be real implies, from (15-53) that the drift speed c of the vortex must be either less than $-\beta_0 R^2$ or greater than zero. In other words, c must lie outside of the range of linear planetary wave speeds [see (15-42b)]. Because k enters the solution in multiplication with the radial distance r, its inverse, $1/k$, can be considered as the width of the vortex:

$$L = \frac{1}{k} = \frac{R}{\sqrt{1 + \beta_0 R^2/c}}. \tag{15-54}$$

The faster the propagation (either eastward or westward), the closer L is to the deformation radius R. Eastward-propagating vortices ($c > 0$) are smaller than R, whereas the westward-propagating ones ($c < 0$) are wider.

The Bessel functions K_m are singular at the origin, and solution (15-52) fails near the vortex center. The situation is remedied by requiring that, in the vicinity of $r = 0$, the function F assumes another form than that used previously, changing the character of the solution there. Here, we shall not consider this solution, called the *modon*, and instead refer the interested reader to Flierl et al. (1980).

Let us now consider disturbances on a zonal jet, such as the meanders of the atmospheric jet stream. Waves and finite-amplitude perturbations propagate zonally at a net speed that is their own drift speed plus the jet average velocity. They thus move away from their region of origin, such as a mountain range, by traveling either upstream or downstream, unless their net speed is about zero. In this case, when the disturbance's own speed c is equal and opposite to the average jet velocity U (positive eastward), the disturbance is stationary and can persist for a much longer time. Typically, the zonal jet flows eastward (jet stream in the atmosphere, the Gulf Stream in the ocean, and prevailing winds on Jupiter at the latitude of the Great Red Spot), and so we take U positive. The mathematical requirement is $c = -U$ (westward), and two cases arise. Either U is smaller than $\beta_0 R^2$ or it is not. For U less than $\beta_0 R^2$, c falls in the range of planetary waves, and the disturbance takes the aspect of a train of planetary waves, giving the jet a meandering character. By virtue of dispersion relation (15-42a), applied to the gravest vertical mode ($i = 1$) and to the zero meridional wave number ($m = 0$), the zonal wavelength is

$$\lambda = \frac{2\pi}{l} = 2\pi R \sqrt{\frac{U}{\beta_0 R^2 - U}}. \tag{15-55a}$$

If the jet velocity varies downstream, the wavelength adjusts locally, increasing with U. If, on the other hand, U exceeds $\beta_0 R^2$, finite-amplitude, isolated disturbances are possible, and the jet may be strongly distorted. The preceding theory suggests the length scale

$$L = \frac{1}{k} = R \sqrt{\frac{U}{U - \beta_0 R^2}}. \tag{15-55b}$$

PROBLEMS

15-1. Derive the one-layer quasi-geostrophic equation

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - \frac{1}{R^2}\psi\right) + J(\psi, \nabla^2\psi) + \beta_0 \frac{\partial \psi}{\partial x} = 0, \tag{15-56}$$

where $R = (gH)^{1/2}/f_0$, from the shallow-water model (4-17) through (4-19) assuming weak surface displacements. How do the waves allowed by these dynamics compare to the planetary waves exposed in Section 15-5?

15-2. Demonstrate the assertion made at the end of Section 15-4 that the time rate of change of available potential energy is proportional to the integral of the product of density perturbation with vertical velocity.

15-3. Elucidate in a rigorous manner the scaling assumptions justifying simultaneously the quasi-geostrophic approximation and the linearization of the equations for the wave analysis. What is the true restriction on vertical displacements?

15-4. Show that the assumption of a rigid upper surface (combined to the assumption of a flat bottom) effectively replaces the external radius of deformation by infinity. Also show that the approximate solutions for the vertical wave number n in Section 15-5 then become exact.

15-5. Explore topographic waves using the quasi-geostrophic formalism on an f-plane ($\beta_0 = 0$). Begin by formulating the appropriate bottom-boundary condition.

Jule Gregory Charney

1917 – 1981

A strong proponent of the idea that intelligent simplifications of a problem are not only necessary to obtain answers but also essential to understand the underlying physics, Jule Gregory Charney was a major contributor to dynamic meteorology. As a student, he studied the instabilities of large-scale atmospheric flows and elucidated the mechanism that is now called baroclinic instability (Chapter 16). His thesis appeared in 1947, and the following year, he published an article outlining quasi-geostrophic dynamics (the material in this chapter). He then turned his attention to numerical weather prediction, an activity envisioned by L. F. Richardson some thirty years earlier. The success of the initial weather simulations in the early 1950s is to be credited not only to J. von Neumann's first electronic computer, but also to Charney's judicious choice of simplified dynamics, the quasi-geostrophic equation. Later on, Charney was instrumental in convincing officials worldwide of the significance of numerical weather predictions, while he also gained much deserved recognition for his work on tropical meteorology, topographic instability, geostrophic turbulence, and the Gulf Stream. Charney applied his powerful intuition to systematic scale analysis. Scaling arguments are now a mainstay in geophysical fluid dynamics. (*Photo from archives of the Massachusetts Institute of Technology.*)