16

Baroclinic Instability

Summary: In a stratified fluid, a baroclinic geostrophic flow is accompanied by tilted density surfaces. A partial reduction of this tilt liberates potential energy that can feed the kinetic energy of disturbances. Thus, the flow may be unstable. The instability mechanism, called baroclinic instability, is at the origin of the large midlatitude cyclones and anticyclones that make our weather so variable.

16-1 CAUSE FOR INSTABILITY

It was pointed out in Section 13-1 that a vertically sheared geostrophic flow in a stratified fluid is accompanied by a tilt of density surfaces. The reason is simple: The geostrophic flow requires a horizontal pressure gradient, which, because of hydrostatics, can only exist if there is a horizontal density gradient. Geostrophic and hydrostatic balances thus combine to maintain a flow (called thermal wind) in equilibrium. However, this equilibrium is not that of least energy, because a reduction of the slope of density surfaces by spreading of the lighter fluid above the heavier fluid would lower the center of gravity and thus the potential energy. Also, it would simultaneously reduce the pressure gradient, the geostrophic flow, and, thus, the kinetic energy. Evidently, the state of rest is that of least energy (minimum potential energy and zero kinetic energy).

In a thermal wind, relaxation of the density distribution and tendency toward the state of rest cannot occur in any direct, spontaneous manner. Such an evolution would require vertical stretching and squeezing of fluid columns, which cannot occur without alteration of potential vorticity.

Friction is capable of modifying potential vorticity, and this is how, under the slow action of friction, a state of thermal wind would decay, eventually bringing the system to rest. But, there is typically a much more dramatic process that operates before the influence of friction becomes noticeable.

Vertical stretching and squeezing of fluid parcels is possible under conservation of potential vorticity if relative vorticity comes into play. As we have seen in Section 12-2, a column of stratified fluid that is stretched vertically develops cyclonic relative vorticity, and one that is squeezed develops anticyclonic vorticity. In a slightly perturbed thermal-wind system, the vertical stretching and squeezing occurring simultaneously at different places generates a pattern of interacting vortices. Under certain conditions, these interactions can increase the initial perturbation, thus forcing the system to evolve away from its original state.

Physically, a first, partial relaxation of the density surfaces liberates a modicum of potential energy, whereas the concomitant stretching and squeezing creates new relative vorticity; the kinetic energy of these new motions can naturally be provided by the potential-energy release. If conditions are favorable, these motions can then contribute to further relaxation of the density field and to stronger vortices. With time, large vortices are formed at the expense of the initial thermal wind. These vortices noticeably increase the amount of velocity shear in the system, greatly enhancing the action of friction. The evolution toward a lower energy level therefore occurs more effectively via the growth of disturbances, transformation from potential into kinetic energy, formation of vortices, and decrease of kinetic energy by friction.

Let us now investigate how a disturbance of a thermal wind can generate a relative-vorticity distribution favorable to growth. For this purpose, a two-fluid idealization, as depicted in Figure 16-1, is sufficient. For the discussion, let us also ignore the beta effect and assign arbitrary north-south and west-east directions for easier identification. With the interface sloping upward to the north, the upper flow is eastward and the lower flow is westward (Figure 16-1). A perturbation of the upper flow will cause some of its parcels to move northward into shallower areas; these will undergo vertical squeezing and thus develop some anticyclonic vorticity (clockwise on the figure). But, because the interface is not a rigid bottom, there is some flexibility, and the interface will be slightly depressed, relieving the upper parcels from some squeezing and at the same time creating a compensating squeeze in the lower layer. Thus lower-layer parcels will also develop anticyclonic vorticity at that location. (Note in passing that a lowering of the interface on the shallower side is also in the direction of a decrease of potential energy.) Elsewhere, the disturbance will cause upper-layer parcels to move in the opposite direction—that is, toward a deeper area. There, vertical stretching will occur, and, again, because the interface is flexible, the stretching in the upper layer will be only partial, the interface will rise somewhat, and a compensating stretching will take place in the lower layer. Thus, parcels in both layers develop cyclonic

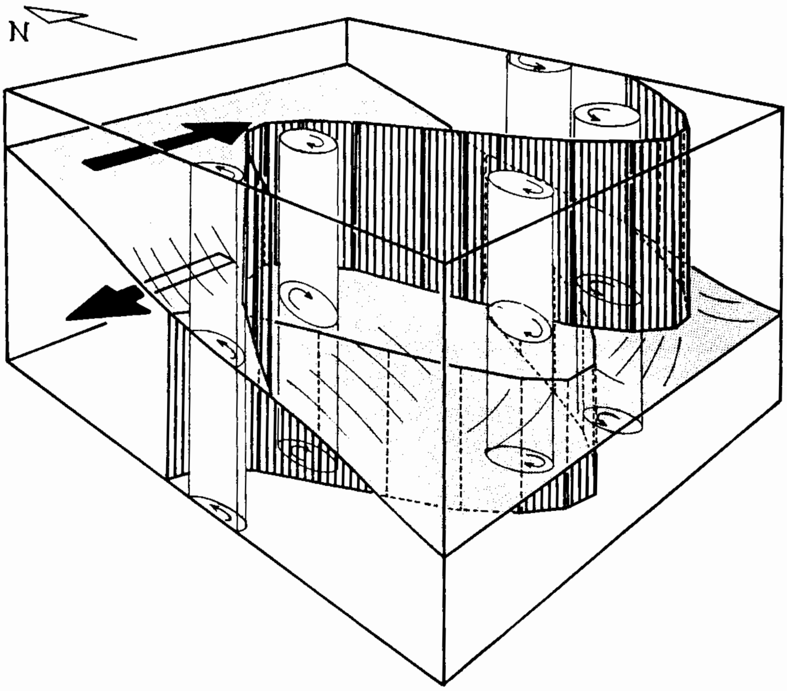


Figure 16-1 Illustration of the baroclinic-instability mechanism in a two-layer system. Lateral displacements in a thermal wind cause vertical stretching and squeezing, which generate relative vorticity. The vortex tubes then act to increase the initial displacements, and the system evolves away from its initial state.

relative vorticity. Note that a lifting of the interface on the deeper side is in the direction of a potential-energy decrease. If the disturbance has some periodicity, alternating northward and southward displacements in the upper layer cause alternating columns of anticyclonic and cyclonic vorticities extending through both layers. Between these columns, the complementary vortical motions create further northward and southward displacements, but because these occur between the crests and troughs, they lead not to growth, but only to a translation of the disturbance. (The mechanism here is identical to that of planetary and topographic waves, discussed in Section 6-6.)

Growth (or decay) will take place if transverse displacements occur in the lower layer as well. The situation is similar: Northward displacements move lower-layer parcels to a thicker environment, vertical stretching occurs, and, because of the flexibility of the interface, some stretching also occurs in the upper layer; on the other hand, southward displacements in the lower layer cause vertical squeezing in both layers and a rise of the interface and columns of alternating relative vorticity are created. If this second set of columns is now in quadrature with the first set (that created by column displacements in the upper layer), the complementary vortical motions of one set will fall at the crests and troughs of the other set, establishing an interaction either favorable or unfavorable to growth. If the spatial phase difference is such that the displacement pattern in one layer is shifted in the direction of the basic flow in that layer, as depicted in Figure 16-1, the vortical motions in one layer induced by the displacements in the other combine to increase the displacements in the former. Thus, the disturbance in each layer amplifies that in the other, and the system evolves away from its thermal-wind equilibrium.

The preceding discussion points to the need of a specific phase arrangement between the displacements in the two layers and emphasizes the role of vorticity generation. A further requirement is necessary for growth: The disturbance must have a wavelength that is neither too short nor too long; it must be such that the vertical stretching and squeezing effectively generates relative vorticity. To show this, let us consider the quasi-geostrophic form of the potential vorticity (15-20), on the f-plane:

$$q = \nabla^2\psi + \frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}\right), \tag{16-1}$$

where ψ is the streamfunction, f the Coriolis parameter, N is the stratification frequency, and ∇^2 is the two-dimensional Laplacian. For a displacement pattern of wavelength L, the first term representing relative vorticity is on the the order of

$$\nabla^2\psi \sim \frac{\Psi}{L^2}, \tag{16-2}$$

where the streamfunction scale Ψ is proportional to the amplitude of the displacements. If the height of the system is H, the second term (representing vertical stretching) scales like

$$\frac{\partial}{\partial z}\left(\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}\right) \sim \frac{f^2\Psi}{N^2H^2} = \frac{\Psi}{R^2}, \tag{16-3}$$

where we have defined the deformation radius $R = NH/f$.

Now, if L is much larger than R, the relative vorticity cannot match the vertical stretching as scaled. This implies that vertical stretching will be inhibited, and the displacements in the layers will tend to be in phase in order to reduce squeezing and stretching of fluid parcels in each layer. On the other hand, if L is much shorter than R, relative vorticity dominates potential vorticity. The two layers become uncoupled, and there is insufficient potential energy to feed a growing disturbance. In sum, displacement wavelengths on the order of the deformation radius are the most favorable to growth.

Because fluctuations are so ubiquitous in nature, an existing flow in thermal-wind balance will continuously be subjected to perturbations. Most of these will have a benign effect, because they do not have the proper phase arrangement or a suitable wavelength. But, sooner or later, a perturbation with both favorable phase and wavelength will occur, prompting the system to evolve irreversibly from its equilibrium state. We therefore conclude that flows in thermal-wind balance are intrinsically unstable. Because their instability process depends crucially on a phase shift with height, the fatal wave must have a baroclinic structure. To reflect this fact, the process has been termed *baroclinic instability*.

It is now believed that the cyclones and anticyclones of our midlatitude weather are manifestations of the baroclinic instability of the atmospheric jet stream. The person who first analyzed the stability of vertically sheared currents (thermal wind) and who demonstrated the relevance of the instability mechanism to our weather is J. G. Charney. (For a short biography, see the end of Chapter 15.) While Charney (1947) performed the stability analysis for a continuously stratified fluid on the beta plane, Eady (1949) did the analysis on the f-plane independently. The comparison between the two theories reveals that the beta effect is a stabilizing influence. Briefly, a change in planetary vorticity (by meridional displacements) is another way to allow vertical stretching and squeezing while preserving potential vorticity. Relative vorticity is then no longer as essential and, in some cases, sufficiently suppressed to render the thermal wind stable to perturbations of all wavelengths.

16-2 LINEAR THEORY

Numerous stability analyses have been published since those of Charney and Eady, exemplifying one aspect or another. Phillips (1954) idealized the continuous vertical stratification to a two-layer system, a case which Pedlosky (1963; 1964) generalized by allowing arbitrary horizontal shear in the basic flow. Barcilon (1964) studied the influence of friction on baroclinic instability by including the effect of Ekman layers, whereas Orlanski (1968; 1969) investigated the importance of non-quasi-geostrophic effects and of a bottom slope. Later, Orlanski and Cox (1973), Gill et al. (1974), and Robinson and McWilliams (1974) confirmed that baroclinic instability is the primary cause of the observed oceanic variability at intermediate scales (tens to hundreds of kilometers).

Here, we shall only present the simplest mathematical model, taken from Eady (1949). The fluid is inviscid ($\nu = 0$), uniformly stratified ($N =$ constant), on the f-plane ($\beta_0 = 0$), over a flat bottom (at $z = 0$), and under a rigid lid (at $z = H$, constant). The basic flow is uniform in the horizontal and uniformly sheared in the vertical without veering:

$$\bar{u} = \frac{U}{H}\, z\,, \qquad \bar{v} = 0\,, \tag{16-4}$$

for $0 \le z \le H$. The velocity is chosen to vanish along the bottom (to inject a certain degree of realism into the model, although the addition of a vertically uniform velocity does not change the stability properties of the flow, the thermal-wind remaining the same), and U is the maximum velocity occurring at the top. The dynamics are chosen to be quasi-geostrophic, and so we introduce a streamfunction ψ and potential vorticity q that obey (15-19) and (15-20):

$$\frac{\partial q}{\partial t} + J(\psi, q) = 0, \tag{16-5a}$$

$$q = \nabla^2\psi + \frac{f^2}{N^2}\frac{\partial^2\psi}{\partial z^2}. \tag{16-5b}$$

From these quantities, the primary physical variables are derived as follows (see Section 15-2):

$$u = -\frac{\partial\psi}{\partial y}, \quad v = +\frac{\partial\psi}{\partial x}, \quad w = -\frac{f}{N^2}\left[\frac{\partial^2\psi}{\partial t\partial z} + J\left(\psi, \frac{\partial\psi}{\partial z}\right)\right], \tag{16-6a}$$

$$p' = \rho_0 f_0 \psi, \quad \rho' = -\frac{\rho_0 f_0}{g}\frac{\partial\psi}{\partial z}. \tag{16-6b}$$

In terms of ψ and q, the basic state (16-4) corresponds to

$$\bar{\psi} = -\frac{U}{H}yz, \quad \bar{q} = 0, \tag{16-7}$$

and thus has a uniform (zero) potential vorticity. As it turns out, this latter property considerably simplifies the mathematical problem but, unfortunately, also renders the model somewhat degenerate (see Section 16-4).

Adding a perturbation ψ' to $\bar{\psi}$ with corresponding perturbation q' to $\bar{q}$, both of infinitesimal amplitudes so that the equations can be linearized, we obtain, from (16-5):

$$\frac{\partial q'}{\partial t} + J(\bar{\psi}, q') + J(\psi', \bar{q}) = 0 \tag{16-8a}$$

$$q' = \nabla^2\psi' + \frac{f^2}{N^2}\frac{\partial^2\psi'}{\partial z^2}. \tag{16-8b}$$

Elimination of q' and replacement of the basic-flow quantities with (16-7) yield a single equation for ψ':

$$\left(\frac{\partial}{\partial t} + \frac{U}{H}z\frac{\partial}{\partial x}\right)\left(\nabla^2\psi' + \frac{f^2}{N^2}\frac{\partial^2\psi'}{\partial z^2}\right) = 0. \tag{16-9}$$

Because this equation has coefficients independent of x, y, and time, a sinusoidal function in those variables is a solution, and we write: $\psi' = \mathrm{Re}[a(z)\exp i(lx + my - \omega t)]$, where $a(z)$ is an unknown, vertically varying amplitude, l and m are horizontal wave-number components (both taken as real and l taken positive), and ω is the frequency. Should this frequency be complex with a positive imaginary part, exponential growth occurs in time, and the wave is deemed unstable. Substitution in (16-9) leads to an equation for the vertical structure of the amplitude function:

$$-i\left[\omega - \frac{lU}{H}z\right]\left[\frac{f^2}{N^2}\frac{d^2a}{dz^2} - (l^2 + m^2)a\right] = 0.$$

The first bracketed quantity cannot vanish at all levels, and we thus require that the other does. The solution is

$$a(z) = A \cosh(nz) + B \sinh(nz), \tag{16-10}$$

where the "wave number" n (not an integer) is defined from

$$n^2 = \frac{N^2}{f^2} (l^2 + m^2) = \frac{R^2}{H^2} (l^2 + m^2). \tag{16-11}$$

The constants of integration, A and B, are to be determined from the boundary conditions in the vertical. These are $w(z = 0) = w(z = H) = 0$, which in terms of the streamfunction, perturbed streamfunction and amplitude function take the successive forms:

$$\frac{\partial^2 \psi}{\partial t \partial z} + J\left(\psi, \frac{\partial \psi}{\partial z}\right) = 0,$$

$$\frac{\partial^2 \psi'}{\partial t \partial z} + J\left(\overline{\psi}, \frac{\partial \psi'}{\partial z}\right) + J\left(\psi', \frac{\partial \overline{\psi}}{\partial z}\right) = 0,$$

$$\left(\omega - \frac{lU}{H} z\right) \frac{da}{dz} + \frac{lU}{H} a = 0, \tag{16-12}$$

at $z = 0$ and $z = H$. The condition at $z = 0$ gives B in terms of A $[B = -(lU/\omega nH)A]$, whereas the condition at $z = H$ yields either $A = 0$ (and there is no perturbation at all) or (the interesting case)

$$\left(\frac{\omega}{lU} - \frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{n^2H^2} - \frac{\coth(nH)}{nH}. \tag{16-13}$$

A real value of ω is possible only if the right-hand side of equation (16-13) is positive, which occurs for $nH \geq 2.399$. Should nH fall below this critical value, two complex conjugate roots are found for ω, one of them with a positive imaginary part, and the perturbation grows exponentially, at least initially until finite-amplitude effects are no longer negligible and energy limitation becomes important. Returning to the definition (16-11) of n, we conclude that all perturbations of horizontal wave number $k = (l^2 + m^2)^{1/2}$ satisfying

$$k < 2.399 \frac{f}{NH}$$

are unstable. The corresponding criterion on the wavelength $\lambda = 2\pi/k$ is

$$\lambda > 2.619 \frac{NH}{f} = 2.619R. \tag{16-14}$$

Thus, all perturbations of wavelength exceeding 2.619 times the deformation radius contribute to take the system away from the equilibrium.

Until nonlinear, finite-amplitude effects become important, the perturbation that distorts the system most is expected to be the one with the greatest initial growth rate, $\text{Im}(\omega)$. For $m = 0$, this wave has $l = k = 1.606/R$, or, equivalently, has a wavelength

$$\lambda = 3.912R \tag{16-15}$$

in the direction of the basic flow. Note that all unstable waves not only grow but also propagate in time. According to (16-13), $\text{Re}(\omega) = lU/2$ when ω is complex, and thus the propagation speed in the direction of the basic flow is $U/2$, or the average velocity of the basic flow.

It is interesting at this point to return to our initial considerations (Section 16-1) and to confirm them with the preceding solution. First and foremost, the fact that the critical wavelength for instability ($2.619R$) and the wavelength of the fastest-growing perturbation ($3.912R$) are both proportional to R with coefficients on the order of unity validates our argument that self-amplification requires a scale on the order of the deformation radius. Physically, it also verifies that the instability process involves a rearrangement of potential vorticity between relative vorticity and vertical stretching. The necessary phase relationship between the transverse displacements of the upper and lower fluids can be checked as follows. Noting the transverse displacements d and using simple kinematics, we can write

$$v' = \frac{\partial d}{\partial t} + \bar{u}\,\frac{\partial d}{\partial x}, \tag{16-16}$$

after linearization. Expressing v' in terms of the streamfunction perturbation ($v' = \partial \psi'/\partial x$) and implementing the wave form $d = \text{Re}[\,b(z) \exp i(lx + my - \omega t]$, we then obtain

$$a(z) = \left(\frac{U}{H}\,z - \frac{\omega}{l}\right) b(z),$$

from which we can deduce the ratio of top and bottom displacements:

$$\frac{b(H)}{b(0)} = \frac{(\omega/lU)\cosh nH - (1/nH)\sinh nH}{\omega/lU - 1}. \tag{16-17}$$

For the fastest growing wavelength ($l = 1.606/R$, $m = 0$, $n = 1.606/H$, $\omega/lU = 0.500 + i\,0.193$), this ratio is

$$\frac{b(H)}{b(0)} = 0.672 - i\,0.741$$

$$= \exp(-\,i\,0.266\pi).$$

Physically, the negative phase angle ($-\,0.266\pi = -\,47.8°$) corresponds to an advance of the top displacement over that on the bottom in the direction of the basic flow. The phase shift is in the sense of that anticipated from the simple physical argument of the previous section. The two extreme displacements are not, however, in quadrature, because the latter argument failed to take into account the continuous nature of the stratification.

From an observational point of view, however, the interest lies in the pressure field, which is proportional to the streamfunction [see (16-6b)]. Within an arbitrary multiplicative constant, which the linear theory cannot determine, the pressure field associated with the fastest growing perturbation can be expressed in terms of the vertical structure of the streamfunction perturbation:

$$a(z) = A\ [\sinh(n(H-z)) + i\ \sinh(nz)]. \qquad (16\text{-}18)$$

From this, we obtain $a(0) = A \sinh(nH)$ and $a(H) = iA \sinh(nH)$ and conclude that the crests and troughs of the pressure pattern at the top lag those of the bottom pattern by a quarter of a wavelength. The amplitude $|a(z)|$ of the pressure disturbance is largest at the top and bottom and reaches a minimum at midheight ($z = H/2$), where the ratio to the maximum value is

$$\frac{|a(H/2)|}{|a(0)|} = \frac{|a(H/2)|}{|a(H)|} = \sqrt{2}\ \frac{\sinh(nH/2)}{\sinh(nH)} = 0.528. \qquad (16\text{-}19)$$

16-3 HEAT TRANSPORT

The qualitative arguments developed in Section 16-1 revolved around the idea that if a flow in thermal-wind equilibrium is unstable, it will seek a level of lower energy and that such an evolution would imply a relaxation of the density surfaces toward static equilibrium. If we now think of the atmosphere, where the heavier fluid is colder air and the lighter fluid warmer air, relaxation implies a flow of warm air toward the colder side (northern side of Figure 16-1) and of cold to the warmer side (southern side of Figure 16-1). In other words, we expect a down-gradient heat flux and, because the atmospheric temperature increases toward the equator, a poleward heat flux. Let us examine what the preceding linear theory predicts.

Relating the density perturbation ρ' to a temperature fluctuation T' via a linearized equation of state [$\rho' = -(\rho_0/T_0)T'$], we express the meridional heat flux carried by the fastest growing disturbance as

$$\overline{v'T'} = -\frac{T_0}{\rho_0}\ \overline{v'\rho'},$$

where v' is the meridional velocity associated with the perturbation and where an overbar denotes an average over one wavelength. With $v' = \partial\psi'/\partial x$ and $\rho' = -(\rho_0 f_0/g)\ \partial\psi'/\partial z$ according to (16-16a) and (16-16b), we obtain, successively,

$$\overline{v'T'} = +\frac{f_0 T_0}{g}\ \overline{\frac{\partial\psi'}{\partial x}\ \frac{\partial\psi'}{\partial z}}$$

$$= \frac{f_0 T_0}{2g}\ \mathrm{Re}\left(ila\ \frac{da^*}{dz}\right)\ e^{2\,\mathrm{Im}(\omega)t}$$

$$= \frac{f_0 T_0}{2g} |A|^2 l n \sinh(nH) \; e^{2\,\mathrm{Im}(\omega)t}, \tag{16-20}$$

which is always positive and, thus, northward as anticipated. It is, moreover, independent of height.

Because the earth is heated in the tropics and cooled at high latitudes, the global heat budget requires a net poleward heat flux in each hemisphere. The flux is carried by both atmosphere and ocean. In the atmosphere, the higher temperatures in the tropics and lower temperatures at high latitudes maintain an overall thermal wind system, which is baroclinically unstable. Vortices on the order of the baroclinic radius of deformation ($R = NH/f \sim 1000$ km) are continuously produced, carry the heat poleward, and tend to relax the thermal-wind structure. The latter, however, is maintained by continuous heating in the tropics and cooling at high latitudes. As a consequence, the large cyclones and anticyclones of our weather are the primary agents of meridional heat transfer in the atmosphere. Without baroclinic instability, they would not exist and weather forecasting would be a much simpler task, but the tropical regions would be much hotter and the polar regions, much colder. Also, the dominance of zonal winds would preclude efficient mixing across latitudes, exacerbating certain problems by severely limiting, for example, the spread of volcanic ash and radioactive fallout. Moreover, less atmospheric variability would imply greatly reduced temperature and moisture contrasts and thus much less precipitation at midlatitudes. All in all, we must concede that baroclinic instability in our atmosphere is very beneficial.

In the ocean, the situation is quite different. The pressure of meridional boundaries prevents thermal-wind-type currents from encircling the globe, and ocean circulation consists of large-scale gyres (Chapter 8). The meridional branches of these gyres, especially the western boundary currents (Gulf Stream in the North Atlantic, Kuroshio in North Pacific), are the main carriers of heat toward high latitudes. This greatly reduces the need for poleward heat transfer by eddies. Baroclinic instability is active in regions of strong currents, such as the Gulf Stream and Kuroshio extensions in the deep ocean but the eddies so created transport little net heat across latitudes.

16-4 MORE-GENERAL CRITERIA

The theory exposed in Section 16-2 is admittedly a very simplified version of baroclinic-instability physics. Since it is not our purpose here to recount the advanced analyses that have been published over the years since the pioneering studies of Charney and Eady (the interested reader will find a survey in the book of Pedlosky, 1987), we will once again turn to integral relations, from which some necessary but not sufficent criteria for instability can be derived. We already used this approach in the study of horizontally sheared currents in homogeneous fluids (Section 7-2) and of vertically sheared currents in nonrotating stratified fluids (Section 11-2). Although a general presentation that would encompass the preceding two situations as well as baroclinic instability could be formulated, it is most instructive to emphasize the conditions necessary for baroclinic instability by basing the analysis on the quasi-geostrophic equation. (Actually, this equation eliminates the Kelvin–Helmholtz instability but not the barotropic instability.) The following derivations are based on the work by Charney and Stern (1962).

We start again with the linearized perturbation equations (16-8a) and (16-8b):

$$\frac{\partial q'}{\partial t} + J(\overline{\psi}, q') + J(\psi', \overline{q}) = 0, \tag{16-21a}$$

$$q' = \nabla^2\psi' + \frac{f_0^2}{N^2}\frac{\partial^2\psi'}{\partial z^2}, \tag{16-21b}$$

where now the basic flow is described by an arbitrary zonal flow $\overline{u}(y, z)$, potentially with both horizontal and vertical shear, with the corresponding streamfunction $\overline{\psi}(y, z)$ and potential-vorticity distribution

$$\overline{q} = \frac{\partial^2\overline{\psi}}{\partial y^2} + \frac{f_0^2}{N^2}\frac{\partial^2\overline{\psi}}{\partial z^2} + \beta_0 y. \tag{16-22}$$

We have restored the beta term for added realism.

Substitution of (16-21b) and (16-22) into (16-21a) yields a single equation for the streamfunction perturbation ψ', which includes nonconstant coefficients depending on the basic flow structure via $\overline{\psi}$. Because those coefficients depend only on y and z, a waveform solution in x and time can be sought: $\psi'(x, y, z, t) = \mathrm{Re}[a(y, z) \exp il(x - ct)]$. The amplitude function $a(y, z)$ must obey

$$\frac{\partial^2 a}{\partial y^2} + \frac{f_0^2}{N^2}\frac{\partial^2 a}{\partial z^2} + \left(\frac{1}{\overline{u} - c}\frac{\partial \overline{q}}{\partial y} - l^2\right) a = 0, \tag{16-23}$$

where $\overline{u} = -\partial\overline{\psi}/\partial y$ and $\overline{q}$ is defined in (16-22).

The upper and lower boundaries are once again idealized to rigid horizontal surfaces, where the vertical velocity must vanish. Repeating the derivations that earlier led to (16-12), we now obtain

$$(\overline{u} - c)\frac{\partial a}{\partial z} - \frac{\partial \overline{u}}{\partial z} a = 0 \tag{16-24}$$

at $z = 0, H$. In the meridional direction, we idealize the domain to a channel of width L between two vertical walls, where the meridional velocity $v' = \partial\psi'/\partial x$ vanishes. We thus impose

$$a = 0 \tag{16-25}$$

at $y = 0, L$.

Multiplying (16-23) by the complex conjugate a^* of a, integrating over the meridional and vertical extents of the domain, performing integrations by parts, and using the preceding boundary conditions, we obtain

$$\int_0^H \int_0^L \left[\left| \frac{\partial a}{\partial y} \right|^2 + \frac{f_0^2}{N^2} \left| \frac{\partial a}{\partial z} \right|^2 + l^2 |a|^2 \right] dy\ dz$$
$$= \int_0^H \int_0^L \frac{1}{\bar{u} - c} \frac{\partial \bar{q}}{\partial y} |a|^2\ dy\ dz$$
$$+ \int_0^L \left[\frac{f_0^2}{N^2} \frac{1}{\bar{u} - c} \frac{\partial \bar{u}}{\partial z} |a|^2 \right]_0^H dy. \qquad (16\text{-}26)$$

The imaginary part of this equation is

$$c_i \left\{ \int_0^H \int_0^L \frac{|a|^2}{|\bar{u} - c|^2} \frac{\partial \bar{q}}{\partial y}\ dy\ dz + \int_0^H \left[\frac{f_0^2}{N^2} \frac{|a|^2}{|\bar{u} - c|^2} \frac{\partial \bar{u}}{\partial z} \right]_0^H dy \right\} = 0. \qquad (16\text{-}27)$$

A necessary condition for instability is that c_i not be zero (so that the disturbance can grow in time). According to (16-27), this implies that the quantity in braces must vanish, and therefore necessary conditions for instability are that either

1. $\partial \bar{q} / \partial y$ changes sign in the domain, or
2. the sign of $\partial \bar{q} / \partial y$ is opposite to that of $\partial \bar{u} / \partial z$ at the top, or
3. the sign of $\partial \bar{q} / \partial y$ is the same as that of $\partial \bar{u} / \partial z$ at the bottom.

A sufficient condition for stability is that none of the above three conditions be met.

Before proceeding, it is worth applying this result to the simple flow analyzed in Section 16-2. With $\bar{u} = Uz/H$ and $\beta_0 = 0$, we have $\bar{q} = 0$ and $\partial \bar{u} / \partial z = U/H$, and (16-27) reduces to

$$c_i \int_0^L \frac{f_0^2 U}{N^2 H} \left[\frac{|a(y, H)|^2}{|U - c|^2} - \frac{|a(y, 0)|^2}{|c|^2} \right] dy = 0,$$

in which the integral is obviously not sign definite. Stability cannot be guaranteed; indeed, our earlier results demonstrate that this flow is always unstable to long waves. Had we instead chosen a weak flow field with no vertical shear at the boundaries [e.g., $\bar{u}(z) = U(3z^2/H^2 - 2z^3/H^3)$] and on the beta plane ($\partial \bar{q} / \partial y \simeq \beta_0$), we would have concluded (after much lengthier mathematics) that this flow is stable to all perturbations. This points to the sensitivity of baroclinic instablity to the type of flow.

Another application of (16-27) is to laterally sheared but vertically uniform flow: $\bar{u}(y)$. Then, the potential-vorticity gradient is $\partial \bar{q} / \partial y = \beta_0 - \partial^2 \bar{u} / \partial y^2$ and (16-27) reduces to

$$c_i \left[H \int_0^L \frac{|a|^2}{|\bar{u} - c|^2} \left(\beta_0 - \frac{\partial^2 \bar{u}}{\partial y^2} \right) dy \right] = 0.$$

Here, we recover the result of barotropic instability obtained in Section (7-2) [see equation (7-15)]. We conclude that the instability conditions stated previously include both barotropic and baroclinic instability criteria.

Charney and Stern (1962) explored the case where $\partial \bar{u}/\partial z$ vanishes at both upper and lower boundaries by assuming a vanishing thermal-wind there (e.g., uniform temperature) and/or taking the limits $H \to \infty$, $\bar{u}(H) \to 0$. Of (16-27), only the first integral remains, and the necessary condition for instability is that $\partial \bar{q}/\partial y$ vanish somewhere in the model, a statement identical in form to—but differing in content from—the barotropic-instability criterion of Section 7-2.

According to Gill et al. (1974), the presence of a bottom slope in the meridional direction modifies the preceding third of the three conditions as follows:

3. The sign of $\partial \bar{q}/\partial y$ is the same as that of $\partial \bar{u}/\partial z - (N^2/f_0) d\bar{b}/dy$ at the bottom $z = \bar{b}(y)$.

Therefore, a bottom slope can act as either a stabilizing or a destabilizing influence. It is generally a stabilizing factor if it creates an ambient potential-vorticity gradient in the same direction as the beta effect (i.e., shallower fluid toward higher latidues; see Figure 6-6) and a destabilizing factor otherwise. However, the theory fails to take into account the zonal topographic gradients that are more common on earth (e.g., the Rocky Mountains in North America and the Mid-Atlantic Ridge along the North Atlantic Ocean).

There exist a number of other studies of baroclinic instability. The interested reader is referred to Gill (1982, Chapter 13) and Pedlosky (1987, Chapter 7).

PROBLEMS

16-1. Demonstrate the assertion made at the end of Section 16-4 that the vertically sheared flow $\bar{u}(z) = U(3z^2/H^2 - 2z^3/H^3)$ in $0 \le z \le H$ is baroclinically stable on the beta plane if U is small.

16-2. Compare the magnitudes of the potential and kinetic energies of the most unstable wave described in Section 16-2.

SUGGESTED LABORATORY DEMONSTRATION

Equipment

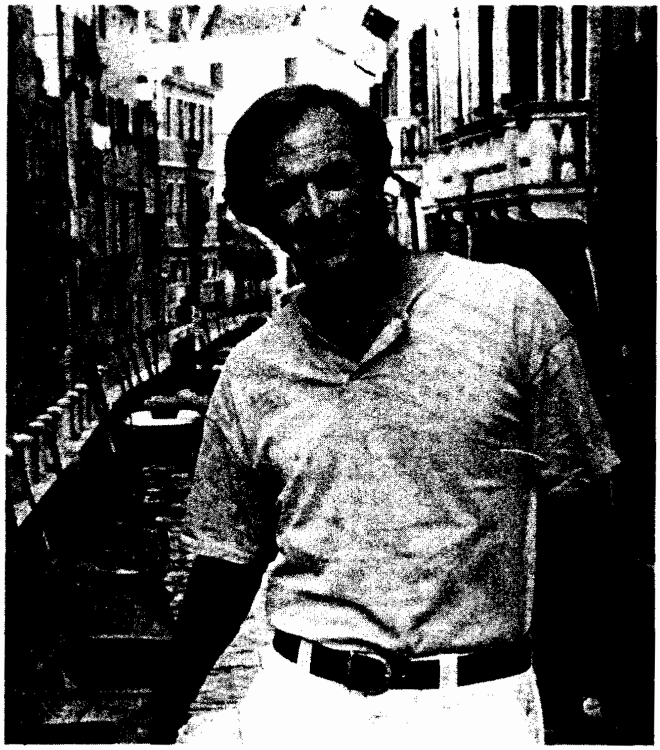
Circular rotating tank (made of a transparent material such as plexiglass), smaller, metallic cyclinder (preferably aluminum or copper), domestic heater (infrared-radiation type), ice cubes, dye.

Experiment

Place the metallic cyclinder in the center of the larger tank and fill the interstitial annulus with water. This fluid will represent the atmosphere of onc hemisphere, between equator and pole. Fill the inner cyclinder with a mixture of water and ice cubes to create the heat sink at the pole. Place the heater on a stationary platform at the level of the tank. When the tank rotates, even warming will ensue, simulating the heat source of the tropical latitudes.

Rotate the tank and wait some time for a stratification and a flow field to be established. A convection cell with rising motions on the outer side and sinking motions on the inner side will be created. The Coriolis effect on this transverse flow will then create a much larger azimuthal flow, cyclonic near the surface and anticyclonic near the bottom. This thermal wind simulates the atmospheric jet stream.

By injecting dye in the water of the annulus and looking through the tank's side, observe the vertical shear of the flow. By looking from above, note the meanders that develop in the azimuthal jet. These result from baroclinic instability and are a crude representation of the cyclones and anticyclones of our midlatitude weather.



Joseph Pedlosky

1938 –

A student of J. G. Charney, Joseph Pedlosky first followed his mentor's footsteps and developed a fascination for baroclinic instability. He quickly became an authority on the subject in his own right, having derived new instability criteria and developed a nonlinear theory for growing baroclinic instabilities in nearly inviscid flow. He also made important contributions to the general theory of rotating stratified fluids, the oceanic thermocline, the Gulf Stream, and the Equatorial Undercurrent. In 1979, Pedlosky published the first treatise on geophysical fluid dynamics, which greatly helped codify the discipline.

Pedlosky's approach to research is first to find a problem that is simple enough to be solved completely, yet physically informative, and then to "worry a great deal about it until I could describe the results to an amateur." This incessant quest for clarity has won him great respect as a scientist and much admiration as a speaker. (*Photo credit: J. Pedlosky.*)