3

The Governing Equations

Summary: The object of this chapter is to establish the equations governing the movement of a stratified fluid in a rotating environment. These equations are immediately simplified in order to be made more pertinent to geophysical flows. Finally, two crucial quantities, the Rossby and Ekman numbers, are identified.

3-1 MOMENTUM EQUATIONS

The considerations developed in the previous chapter enable us to state Newton's law in a rotating framework. For a fluid, the law *mass times acceleration equals the sum of forces* is better stated per unit volume, with density replacing mass. Thus, by virtue of (2-19a) through (2-19c), we can write

$$x: \quad \rho\left(\frac{du}{dt} + f_* w - f v\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau^{xx}}{\partial x} + \frac{\partial \tau^{xy}}{\partial y} + \frac{\partial \tau^{xz}}{\partial z} \tag{3-1}$$

$$y: \quad \rho\left(\frac{dv}{dt} + f u\right) = -\frac{\partial p}{\partial y} + \frac{\partial \tau^{xy}}{\partial x} + \frac{\partial \tau^{yy}}{\partial y} + \frac{\partial \tau^{yz}}{\partial z} \tag{3-2}$$

$$z: \quad \rho\left(\frac{dw}{dt} - f_* u\right) = -\frac{\partial p}{\partial z} - \rho g + \frac{\partial \tau^{xz}}{\partial x} + \frac{\partial \tau^{yz}}{\partial y} + \frac{\partial \tau^{zz}}{\partial z}, \tag{3-3}$$

where $f = 2\Omega \sin \varphi$ is the Coriolis parameter, $f_* = 2\Omega \cos \varphi$ is the reciprocal Coriolis parameter, ρ is density, p is pressure, g is the gravitational acceleration, and the τ terms represent the normal and shear stresses due to friction.

That the pressure force is equal and opposite to the pressure gradient and that the viscous force involves the derivatives of a stress tensor should be familiar to the student who has had an introductory course in fluid mechanics. The effective gravitational force (sum of true gravitational force and the centrifugal force; see Section 2-3) is ρg per unit volume and is directed vertically downward.

Because the acceleration is not measured by the rate of change in velocity at a fixed location but by the change in velocity of a fluid particle as it moves with the flow, the time derivatives in the acceleration components, du/dt, dv/dt, and dw/dt, consist of both the local time rate of change and of the so-called advective terms:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}. \tag{3-4}$$

Finally, because the x-, y-, and z-axes are everywhere aligned with the local eastward, northward, and upward directions, our chosen frame of reference forms a curvilinear coordinate system, and curvature terms enter the equations. To be exact, (3-1) through (3-3) must be augmented as follows:

$$\rho\left(\frac{du}{dt} - \frac{uv \tan \varphi}{r} + \frac{uw}{r} + f_* w - fv\right) = -\frac{\partial p}{\partial x} + F_x$$

$$\rho\left(\frac{dv}{dt} + \frac{u^2 \tan \varphi}{r} + \frac{vw}{r} + fu\right) = -\frac{\partial p}{\partial y} + F_y$$

$$\rho\left(\frac{dw}{dt} - \frac{u^2 + v^2}{r} - f_* u\right) = -\frac{\partial p}{\partial z} - \rho g + F_z,$$

where φ is the latitude and r is the distance to the center of the earth (or planet or star). The components F_x, F_y, and F_z of the frictional force have complicated expressions and need not be reproduced here. For a detailed development of these equations, the reader is referred to Chapter 4 of the book by Gill (1982).

For simplicity in the exposition of the basic principles of geophysical fluid dynamics, we shall neglect here the extraneous curvature terms. To justify doing so, we restrict our attention to length scales L substantially shorter than the radius of the earth (or other planet or star): $L \ll r$. On the earth, a length scale not exceeding 1000 km is usually acceptable. The neglect of the curvature terms is in some ways analogous to the distortion introduced by mapping the curved earth's surface onto a plane.

3-2 OTHER GOVERNING EQUATIONS

Equations (3-1) through (3-3) can be viewed as three equations providing the three velocity components. But, they introduce two additional quantities, namely, the pressure p and the density ρ. Hence, additional equations are required.

3-2-1 Mass Conservation

A necessary statement is that mass be conserved. That is, the imbalance between convergence and divergence in the three spatial directions must translate into a local compression or expansion of the fluid:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \tag{3-5}$$

This equation, often called the *continuity equation*, is classical in traditional fluid mechanics and is not further discussed here. For a detailed derivation, the reader is referred to Batchelor (1967), Fox and McDonald (1992), or any other introductory text. Note that the spherical geometry introduces additional curvature terms, which we again neglect to be consistent with our previous restriction to length scales substantially shorter than the global scale.

3-2-2 Energy Equation

One additional equation is still required to complete the description of the system. To the rescue comes the energy equation that states that the internal energy of a fluid parcel obeys a balanced budget. For most geophysical fluid applications, in which fluid parcels never undergo tremendous changes in temperature and entropy, this energy budget can be considerably simplified.

The first law of thermodynamics states that the internal energy gained by a parcel of matter is equal to the heat it receives minus the mechanical work it performs. Per unit mass and unit time, we write

$$\frac{de}{dt} = Q - p\,\frac{dv}{dt}, \tag{3-6}$$

where the $e = C_v T$ is the internal energy per mass, Q is the rate of heat gained per unit mass, and $v = 1/\rho$ is the specific volume. In the expression for e, C_v is the heat capacity at constant volume and T is the absolute temperature. Because geophysical fluids do not usually contain internal heat sources, the heat gained by a parcel is the result of lateral diffusion. Using the Fourier law, we write $\rho Q = k\nabla^2 T$, where k is the thermal conductivity of the fluid. Equation (3-6) then becomes

$$\rho C_v\,\frac{dT}{dt} - \frac{p}{\rho}\,\frac{d\rho}{dt} = k\nabla^2 T,$$

which, by elimination of $d\rho/dt$ with the continuity equation (3-5), can also be written as

$$\rho C_v\,\frac{dT}{dt} + p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = k\nabla^2 T. \tag{3-7}$$

This is the energy equation. Because it introduces an additional variable—namely, the temperature T—closing the system of equations requires yet another equation.

3-2-3 Equation of State

For any fluid, the density is a function of pressure and temperature: $\rho = \rho(p, T)$. The particular form of this *equation of state* simply tells how density increases under compression and varies with temperature. To go further, we need to distinguish between air and water.

Dry air in the atmosphere behaves approximately as an ideal gas, and so we write

$$\rho = \frac{p}{RT}, \tag{3-8}$$

where $R = C_p - C_v$ and C_p is the heat capacity at constant pressure ($C_p = 1005$ $\mathrm{m^2/s^2 \cdot K}$, $C_v = 718$ $\mathrm{m^2/s^2 \cdot K}$, and $R = 287$ $\mathrm{m^2/s^2 \cdot K}$ at ordinary temperatures and pressures).

Because water is nearly incompressible, its density can be considered as independent of pressure. On the other hand, the density of seawater is affected not only by temperature (warmer waters are lighter), but also by salinity (saltier waters are heavier). In first approximation, a linear equation of state can be adopted:

$$\rho = \rho_0 [1 - \alpha(T - T_0) + \beta(S - S_0)], \tag{3-9}$$

where T is the temperature and S the salinity (in grams of salt per kilogram of seawater, i.e., in parts per mil, denoted by ‰). The constants ρ_0, T_0, and S_0 are reference values of density, temperature, and salinity, respectively, whereas α is the coefficient of thermal expansion and β is called, by analogy, the coefficient of saline contraction. (The latter expression is a misnomer, since salinity increases density not by contraction of the water but by the added mass of dissolved salt.) Typical seawater values are $\rho_0 = 1028$ $\mathrm{kg/m^3}$, $T_0 = 10°\mathrm{C} = 283$ K, $S_0 = 35‰$, $\alpha = 1.7 \times 10^{-4}\ \mathrm{K^{-1}}$, $\beta = 7.6 \times 10^{-4}$, and $C_v = 4000$ $\mathrm{m^2/s^2 \cdot K}$.

This introduces an additional variable, namely, salinity. A local salt budget yields the following equation:

$$\frac{dS}{dt} = \kappa_S \nabla^2 S, \tag{3-10}$$

which states simply that seawater parcels conserve their salt content except in the face of diffusion. The coefficient κ_S is the coefficient of salt diffusion.

Our set of governing equations is now complete. For air (or any ideal gas), there are six variables (u, v, w, p, ρ, and T) for which we have three momentum equations, (3-1) through (3-3), a continuity equation, (3-5), an energy equation, (3-7), and an equation of state, (3-8). For seawater, there are seven variables (u, v, w, p, ρ, T, and S) for which we have the same momentum, continuity, and energy equations, an equation of state, (3-9), and a salt equation, (3-10). For other liquids or if salinity variations are unimportant, the last equation can be ignored, and the salinity dependency can be dropped from the equation of state.

3-3 THE BOUSSINESQ APPROXIMATION

The equations established in the previous sections already contain numerous simplifying approximations (such as the use of local Cartesian coordinates). Yet, as they stand, they are still too complicated for the purpose of geophysical fluid dynamics. Further simplifications can be obtained by the so-called Boussinesq approximation without appreciable loss of accuracy.

In most geophysical systems, the fluid density varies, but not greatly, around a mean value. For example, the average temperature and salinity in the ocean are $T = 4°\text{C}$ and $S = 34.7‰$, to which corresponds a density $\rho = 1028\ \text{kg/m}^3$ at surface pressure; variations in density within one ocean basin rarely exceed $3\ \text{kg/m}^3$. Even in estuaries where fresh river waters ($S = 0‰$) eventually turn into salty seawaters ($S = 34.7‰$), the relative density difference is less than 2%. By contrast, the air in the atmosphere becomes more and more rarefied with altitude, and the density varies from a maximum at ground level to nearly zero at great heights, thus covering a 100% range of variations. Most of the density changes, however, can be attributed to isostatic pressure effects, leaving only a moderate variability due to buoyancy effects. Furthermore, weather patterns are confined to the lowest layer, the troposphere (approximately 10 km thick), within which the density variations responsible for the winds are usually no more than 5%. The situation is obviously somewhat uncertain on other planets with a known fluid layer (Jupiter and Neptune, for example) and on the sun.

So, it appears justifiable in most instances to assume that the fluid density, ρ, does not depart much from a mean reference value, ρ_0. We thus write

$$\rho = \rho_0 + \rho'(x, y, z, t), \qquad \rho' \ll \rho_0, \tag{3-11}$$

where ρ', the variation caused by the existing stratification and/or the fluid motions, is small compared to the reference value ρ_0. Armed with this assumption, we proceed to simplify the governing equations.

The continuity equation, (3-5), can be expanded as follows:

$$\rho_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \rho'\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \left(\frac{\partial \rho'}{\partial t} + u\,\frac{\partial \rho'}{\partial x} + v\,\frac{\partial \rho'}{\partial y} + w\,\frac{\partial \rho'}{\partial z}\right) = 0.$$

Geophysical flows indicate that relative variations of density in time and space are not larger than—and usually much less than—the relative variations of the velocity field. This implies that the terms in the third group are on the same order as—if not much less than—those in the second. Now, terms in this second group are always much less than those in the first, since $\rho' \ll \rho_0$. Therefore, only that first group of terms needs to be retained, and we write

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{3-12}$$

Physically, this statement means that conservation of mass has become conservation of volume. This is to be expected since, density being nearly uniform, volume is a good proxy for mass. Another implication is the elimination of sound waves.

The x- and y-momentum equations (3-1) and (3-2), being similar, can be treated simultaneously. There, ρ occurs as a factor only in front of the left-hand side. So, wherever ρ' occurs, ρ_0 is there and dominates. It is thus safe to neglect ρ' next to ρ_0 in that pair of equations. Then, the assumption of a Newtonian fluid (viscous stresses proportional to velocity gradients), with the use of the reduced continuity equation, (3-12), permits us to write the components of the stress tensor as

$$\tau^{xx} = \mu\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right), \quad \tau^{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \tau^{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right),$$

$$\tau^{yy} = \mu\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right), \quad \tau^{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),$$

$$\tau^{zz} = \mu\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial z}\right),$$

where μ is called the coefficient of dynamic viscosity. A subsequent division by ρ_0 and the introduction of the kinematic viscosity $\nu = \mu/\rho_0$ yield

$$\frac{du}{dt} + f_* w - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\nabla^2 u \tag{3-13}$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\nabla^2 v, \tag{3-14}$$

where the Laplace operator is defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Next is the z-momentum equation, (3-3). There, ρ appears as a factor not only in front of the left-hand side, but also in a product with g on the right. On the left, it is safe to neglect ρ' in front of ρ_0 for the same reason as above, but on the right it is not. Indeed, the term ρg accounts for the weight of the fluid, which, as we know, causes an increase of pressure with depth (or, a decrease of pressure with height, depending on whether we think of the ocean or atmosphere). With the ρ_0 part of the density goes a hydrostatic pressure p_0, a function only of z:

$$p = p_0(z) + p'(x, y, z, t) \tag{3-15}$$

$$p_0(z) = P_0 - \rho_0 g z, \tag{3-16}$$

so that $dp_0/dz = -\rho_0 g$, and the equation at this stage reduces to

$$\frac{dw}{dt} - f_* u = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{\rho' g}{\rho_0} + \nu\nabla^2 w, \tag{3-17}$$

after a division by ρ_0 for convenience. No further simplification is warranted because the remaining ρ' term no longer falls in the shadow of a neighboring term proportional to ρ_0.

Note that the hydrostatic pressure $p_0(z)$ can be subtracted from p in the reduced momentum equations, (3-13) and (3-14), because it has no derivatives with respect to x and y.

The treatment of the energy equation, (3-7), requires some care. First, continuity of volume, (3-12), eliminates the middle term, leaving

$$\rho C_v \frac{dT}{dt} = k\nabla^2 T.$$

Next, the factor ρ in front of the first term can be replaced by ρ_0, for the same reason as it was done in the momentum equations. Defining the heat diffusivity $\kappa_T = k/\rho_0 C_v$, we then obtain

$$\frac{dT}{dt} = \kappa_T \nabla^2 T, \tag{3-18}$$

which is isomorphic to the salt equation, (3-10).

For seawater, the two equations (3-10) for salinity and (3-18) for temperature combine to determine the evolution of density. A simplification results if it may be assumed that the salt and heat diffusivities, κ_S and κ_T, can be taken as equal. If diffusion is primarily governed by molecular processes, this assumption cannot be made. In fact, a substantial difference between the rates of salt and heat diffusion is responsible for peculiar small-scale features, such as salt fingers, which are studied in the discipline called *double diffusion* (Turner, 1973, Chapter 8). But, molecular diffusion generally affects only small-scale processes (up to a meter or so), whereas turbulence regulates diffusion on larger scales. In turbulence, efficient diffusion is accomplished by eddies, which, obviously, mix salt and heat at equal rates. As a result, the values of diffusivity coefficients in most geophysical applications should not be taken as those of molecular diffusion; instead, they should be taken much larger and equal to each other. Such a turbulent diffusion coefficient, also called *eddy diffusivity*, is typically expressed as the product of a turbulent eddy velocity by a mixing length (Tennekes and Lumley, 1972), and although there exists no single value applicable to all situations, the value $\kappa_S = \kappa_T = 10^{-2}$ m^2/s^{-1} is frequently adopted. Noting $\kappa = \kappa_S = \kappa_T$ and combining equations (3-10) and (3-18) with the equation of state (3-9), we obtain

$$\frac{d\rho'}{dt} = \kappa\nabla^2\rho', \tag{3-19}$$

where $\rho' = \rho - \rho_0$ is the density variation. In sum, the energy equation has turned into a density equation.

For air, the treatment of the energy equation is much more subtle, and the reader interested in a rigorous discussion is referred to the article by Spiegel and Veronis (1960). Here, for the sake of simplicity, we will limit ourselves to suggestive arguments. First, we recognize that after having replaced mass by volume conservation, we have vowed to discard all volume changes experienced by air parcels. This eliminates adiabatic heating and cooling and amounts to the neglect of density variations induced by pressure changes. Then, according to the equation of state, (3-8), it follows that density is a function of temperature only. For weak departures from a reference state (distinct for air parcels at different vertical levels), the relationship can be further linearized, again turning the energy equation, (3-18), into (3-19). A discussion of the compressibility of air under pressure changes and of the accompanying adiabatic temperature variations can be found in Section 9-3.

In summary, the Boussinesq approximation, rooted on the assumption that the density does not depart much from a mean value, has allowed us to replace the exact density ρ by its reference value ρ_0 everywhere, except in front of the gravitational acceleration and in the energy equation, which has become an equation governing density variations.

At this point, since the original variables ρ and p no longer appear in the equations, it is customary to drop the primes from ρ' and p' without risk of ambiguity. So, from here on, the variables ρ and p will be used exclusively to denote the perturbation density and perturbation pressure. This perturbation pressure is sometimes called the *dynamic pressure*, because it is usually a main contributor to the flow field.

3-4 FURTHER SIMPLIFICATIONS

Simplifications beyond the Boussinesq approximation are possible. But, these require a discussion of orders of magnitude. So, let us introduce a scale for every variable. By *scale*, we mean a dimensional constant of dimensions identical to that of the variable and having a numerical value representative of the values of that same variable. Table 3-1 provides illustrative scales for all the variables of interest here. Obviously, scale values will vary with every application, and the values listed in Table 3-1 are only suggestive. Even so, the conclusions drawn from the use of these particular values stand in the vast majority of cases. If doubt arises in a specific situation, we can always redo the following scale analysis.

In the construction of Table 3-1, we were careful to satisfy the criteria of geophysical fluid dynamics outlined in Chapter 1; for the time scale,

$$T \gtrsim \frac{1}{\Omega}, \tag{3-20}$$

and for the velocity and length scales,

$$\frac{U}{L} \lesssim \Omega. \tag{3-21}$$

In constructing Table 3-1, we did not find it necessary to discriminate between the two horizontal directions, thus assigning the same length scale L to both coordinates and the same velocity scale U to both velocity components. The same cannot be said of the vertical direction. Indeed, geophysical flows are typically confined to domains that are much wider than they are thick. The atmospheric layer that determines our weather is

TABLE 3-1 TYPICAL SCALES OF ATMOSPHERIC AND OCEANIC FLOWS

Variable	Scale	Unit	Atmospheric value	Oceanic value
x, y	L	m	100 km = 10^5 m	10 km = 10^4 m
z	H	m	1 km = 10^3 m	100 m = 10^2 m
t	T	s	$\geq \frac{1}{2}$ day $\simeq 4 \times 10^4$ s	$\gtrsim 1$ day $\simeq 9 \times 10^4$ s
u, v	U	m/s	10 m/s	0.1 m/s
w	W	m/s		
p	P	kg/m·s^2		
ρ	$\Delta\rho$	kg/m^3	1% of ρ_0	0.1% of ρ_0

only about 10 km thick, yet cyclones and anticyclones spread over thousands of kilometers. Similarly, ocean currents, generally confined to the upper hundred meters of the water column, may extend over tens of kilometers (or more, up the width of the ocean basin). It follows that, for large-scale motions,

$$H \ll L, \tag{3-22}$$

and we expect W to be vastly different from U.

The continuity equation in its reduced form, (3-12), contains three terms of respective orders of magnitude:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

$$\frac{U}{L}, \quad \frac{U}{L}, \quad \frac{W}{H}.$$

We must examine three cases: W/H is much less than, on the order of, or much greater than U/L. The third case must be ruled out. Indeed, if $W/H \gg U/L$, the equation reduces in first approximation to $\partial w/\partial z = 0$, which implies that w is constant in the vertical; because of a bottom somewhere, flow must be supplied by lateral convergence, the terms $\partial u/\partial x$ and/or $\partial v/\partial y$ cannot be neglected, and thus w must be much smaller than we thought. In the first case, the leading balance is $\partial u/\partial x + \partial v/\partial y = 0$, which implies that convergence in one horizontal direction must be neutralized by a divergence in the other horizontal direction. This is very possible. The middle case, when W/H is on the order of U/L, implies a three-way balance, which is also acceptable. In summary, the vertical-velocity scale must be constrained by

$$W \lesssim \frac{H}{L} U \tag{3-23}$$

and, by virtue of (3-22),

$$W \ll U. \tag{3-24}$$

In other words, large-scale geophysical flows are shallow ($H \ll L$) and almost two-dimensional ($W \ll U$).

Let us now consider the x-momentum equation in its Boussinesq form (3-13). After expansion of the material derivative via (3-4), terms scale sequentially as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+f_{*}w\;-\;fv\;=-\frac{1}{\rho_0}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial x^2}+\nu\frac{\partial^2 u}{\partial y^2}+\nu\frac{\partial^2 u}{\partial z^2},$$

$$\frac{U}{T},\quad \frac{U^2}{L},\quad \frac{U^2}{L},\quad \frac{WU}{H},\quad \Omega W,\quad \Omega U,\qquad \frac{P}{\rho_0 L},\quad \frac{\nu U}{L^2},\quad \frac{\nu U}{L^2},\quad \frac{\nu U}{H^2}.$$

The previous remark immediately shows that the fifth term (ΩW) is always much smaller than the sixth (ΩU) and can thus be safely neglected. (Note, however, that near the equator, where f goes to zero while f_{*} reaches its maximum, the simplification may be invalidated. If this is the case, a re-examination of the scales is warranted, although the fifth term is likely to remain much smaller than some other terms, such as the pressure gradient. Otherwise, the f_{*} term must be retained after all, but because such a situation is exceptional, we will dispense with the f_{*} term here.) Next, we note that the last term is much greater than the two preceding it, which we then neglect. Similar simplifications can be made to the y-momentum equation (3-14).

A greater number of simplifications arises from the vertical momentum equation (3-17). After expansion of the material derivative via (3-4), the terms scale sequentially as

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}-\;f_{*}u=-\frac{1}{\rho_0}\frac{\partial p}{\partial z}-\;\frac{g\rho}{\rho_0}\;+\nu\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial z^2}$$

$$\frac{W}{T},\quad \frac{UW}{L},\quad \frac{UW}{L},\quad \frac{W^2}{H},\quad \Omega U,\qquad \frac{P}{\rho_0 H},\quad \frac{g\Delta\rho}{\rho_0},\quad \frac{\nu W}{L^2},\quad \frac{\nu W}{L^2},\quad \frac{\nu W}{H^2}.$$

The first term (W/T) cannot exceed ΩW, which is itself much less than ΩU, by virtue of (3-20) and (3-24). The next three terms are also much smaller than ΩU, this time because of (3-21), (3-23), and (3-24). Thus, the first four terms can all be neglected in front of the fifth. But, this fifth term is itself quite small. Its ratio to the second term on the right-hand side yields

$$\frac{\rho_0 \Omega U}{g\Delta\rho},$$

which, according to the numbers in Table 3-1, together with $\Omega \simeq 10^{-4}\ \mathrm{s}^{-1}$ and $g \simeq 10\ \mathrm{m/s^2}$, ranges from 10^{-1} (atmosphere) to 10^{-3} (ocean).

Then, since $L \gg H$, the last term is again much greater than the two preceding it, which we then neglect. This last term is itself extremely small. Indeed, although vertical friction is retained in (3-10) and (3-11), it cannot dominate the Coriolis force in geophysical flows, implying

$$\frac{\nu U}{H^2} \lesssim \Omega U,$$

which, when W is substituted for U, yields

$$\frac{\nu W}{H^2} \lesssim \Omega W \ll \Omega U.$$

Thus, the last term on the right-hand side of the equation is much less than the fifth term on the left, which was already found to be small. In summary, the vertical momentum balance reduces to the simple hydrostatic relation

$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{g\rho}{\rho_0}.$$

Recall that the pressure p is already a small perturbation to a much larger pressure, itself in hydrostatic balance. Therefore, large-scale geophysical flows tend to be fully hydrostatic even in the presence of substantial motions. Looking back, we note that the main reason behind this simple result is the strong geometric disparity of geophysical flows ($H \ll L$).

Finally, it remains to analyze the density equation, (3-19). The only obvious simplification here is in the diffusion term. Since $H \ll L$, vertical diffusion by far dominates horizontal diffusion, leaving

$$\frac{d\rho}{dt} = \kappa \frac{\partial^2 \rho}{\partial z^2}.$$

3-5 RECAPITULATION OF THE EQUATIONS GOVERNING GEOPHYSICAL FLOWS

The previous Boussinesq approximation and scale analysis have simplified our governing equations drastically. Let us recall the five equations:

x-momentum: $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}, \quad (3\text{-}25)$$

y-momentum: $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}, \quad (3\text{-}26)$$

z-momentum: $$0 = -\frac{\partial p}{\partial z} - \rho g, \quad (3\text{-}27)$$

continuity: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (3\text{-}28)$$

density: $$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \kappa\frac{\partial^2 \rho}{\partial z^2}, \quad (3\text{-}29)$$

where $f = 2\Omega \sin \varphi$ and where ρ_0, g, ν, and κ are constants. These five equations for the five variables u, v, w, p, and ρ form the basis of geophysical fluid dynamics.

3-6 THE ROSSBY AND EKMAN NUMBERS

The scaling analysis of Section 3-4 was developed to justify the neglect of some small terms. But this does not necessarily imply that the remaining terms are equally large. We now wish to estimate the relative sizes of those terms that have been retained.

The terms of the horizontal momentum equations in their last form, (3-25) and (3-26), scale sequentially as

$$
\begin{array}{cccccccc}
\dfrac{\partial u}{\partial t} + & u\dfrac{\partial u}{\partial x} + & v\dfrac{\partial u}{\partial y} + & w\dfrac{\partial u}{\partial z} - & fv & = & -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial x} + & \nu\dfrac{\partial^2 u}{\partial z^2}, \\[2ex]
\dfrac{\partial v}{\partial t} + & u\dfrac{\partial v}{\partial x} + & v\dfrac{\partial v}{\partial y} + & w\dfrac{\partial v}{\partial z} + & fu & = & -\dfrac{1}{\rho_0}\dfrac{\partial p}{\partial y} + & \nu\dfrac{\partial^2 v}{\partial z^2}, \\[2ex]
\dfrac{U}{T}, & \dfrac{U^2}{L}, & \dfrac{U^2}{L}, & \dfrac{WU}{H}, & \Omega U, & & \dfrac{P}{\rho_0 L}, & \dfrac{\nu U}{H^2}.
\end{array}
$$

By definition, geophysical fluid dynamics treats those motions in which rotation is an important factor. Thus, the term ΩU is central to the preceding sequence. A division by ΩU, to measure the importance of all other terms relative to the Coriolis term, yields the following sequence of dimensionless ratios:

$$
\frac{1}{\Omega T}, \quad \frac{U}{\Omega L}, \quad \frac{U}{\Omega L}, \quad \frac{WL}{UH} \cdot \frac{U}{\Omega L}, \quad 1, \quad \frac{P}{\rho_0 \Omega L U}, \quad \frac{\nu}{\Omega H^2}.
$$

The first ratio,

$$
Ro_T = \frac{1}{\Omega T}, \tag{3-30}
$$

is called the *temporal Rossby number*. It compares the local time rate of change of the velocity to the Coriolis force and is on the order of unity or less, as it has been repeatedly stated [see (3-20)]. The next number,

$$
Ro = \frac{U}{\Omega L}, \tag{3-31}
$$

which compares advection to Coriolis force, is called the *Rossby number*[1] and is fundamental in geophysical fluid dynamics. Like its temporal analogue Ro_T, it is at most on the order of unity [see (3-21)]. As a general rule, the characteristics of geophysical flows vary greatly with the values of the Rossby numbers. The next number is the product of the Rossby number by WL/UH, which is on the order of one or less by virtue of (3-23). (It is shown in Chapter 9 that the ratio WL/UH is generally on the order of the Rossby number itself.)

The last number, which measures the relative importance of friction,

[1] See biographical note at the end of this chapter.

$$Ek = \frac{\nu}{\Omega H^2}, \tag{3-32}$$

is called the *Ekman number*. For geophysical flows, this number is exceedingly small; it is very small even in the laboratory, where H is much more modest. [Typical experimental values are $\Omega = 4\ s^{-1}$, $H = 20$ cm, and ν(water) $= 10^{-6}\ m^2/s$, yielding $Ek = 6 \times 10^{-6}$.] Fluid turbulence at subgeophysical scales (small eddies and billows) can act as a dissipative mechanism, thus calling for the substitution of the molecular viscosity by a much larger *eddy viscosity* (Tennekes and Lumley, 1972). Yet, with an eddy viscosity as large as $10^{-2}\ m^2/s$, the Ekman number remains small. (Take $\Omega = 7.3 \times 10^{-5}\ s^{-1}$, $H = 100$ m, and $\nu = 10^{-2}\ m^2/s$ to get $Ek = 1.4 \times 10^{-2}$.) The reason for retaining the frictional force will become clear in Chapter 5, where it is shown that vertical friction creates a very important boundary layer.

In nonrotating fluid dynamics, it is customary to compare inertial and frictional forces by defining the Reynolds number, Re. Above, inertial and frictional forces were not compared to each other; instead, each was compared to the Coriolis force, yielding the Rossby and Ekman numbers, respectively. There exists a simple relationship between the three numbers:

$$Re = \frac{UL}{\nu} = \frac{U}{\Omega L} \cdot \frac{\Omega H^2}{\nu} \cdot \frac{L^2}{H^2} = \frac{Ro}{Ek}\left(\frac{L}{H}\right)^2. \tag{3-33}$$

Since the Rossby number is on the order of unity or slightly less, but the Ekman number and the geometric ratio H/L are much smaller than unity, the Reynolds number of geophysical flows is usually extremely large. The flows are turbulent, and this is why an eddy viscosity must replace molecular viscosity in the momentum equations.

The remaining dimensionless ratio, $P/\rho_0 \Omega LU$, relates the strength of the pressure force to the Coriolis force. Since the Coriolis force is by assumption an important contribution, if not the dominant one, it is natural to think that pressure forces within the flow will develop to reach a level at which they are able to counteract, at least partially, the Coriolis force so that the equations of motion can be met. This principle naturally provides a scale for the dynamic pressure:

$$P = \rho_0 \Omega LU. \tag{3-34}$$

For typical geophysical flows, this pressure is much smaller than the basic hydrostatic pressure due to the weight of the fluid.

PROBLEMS

3-1. A laboratory tank consists of a cylindrical container 30 cm in diameter, filled at rest by 20 cm of fresh water and then spun at 30 rpm. After a state of solid-body rotation is achieved, what is the difference in water level between the rim and the center? How does this difference compare with the minimum depth at the center?

3-2. From the weather chart in today's edition of your local newspaper, identify the horizontal extent of a major atmospheric feature and find the forecast wind speed. From these numbers, estimate the Rossby number of the weather pattern. What do you conclude about the importance of the Coriolis force? (*Hint*: When converting latitudinal and longitudinal differences in kilometers, use the earth's mean radius, 6371 km.)

3-3. Using the scale given in (3-34), compare the dynamic pressure induced by the Gulf Stream (speed = 1 m/s, width = 40 km, and depth = 500 m) with the main hydrostatic pressure due to the weight of the same water depth. Also, convert the dynamic-pressure scale to its equivalent height of hydrostatic pressure (head). What can you infer about the possibility of measuring oceanic dynamic pressures by a pressure gauge?

Carl-Gustaf Arvid Rossby

1898 – 1957

A Swedish meteorologist, Carl-Gustaf Rossby is credited with most of the fundamental principles on which geophysical fluid dynamics rests. Among other contributions, he left us the concepts of planetary waves (Chapter 6), radius of deformation (Chapter 6), and geostrophic adjustment (Chapter 12). However, the dimensionless number that now bears his name was first introduced by the Soviet scientist I. A. Kibel' in 1940. Inspiring to young scientists, whose company he constantly sought, Rossby viewed scientific research as an adventure and a challenge. His accomplishments are marked by a broad scope and what he liked to call the *heuristic approach*, that is, the search for a useful answer without unnecessary complications. During a number of years spent in the United States, he established the meteorology departments at MIT and the University of Chicago. He later returned to his native Sweden to become the director of the Institute of Meteorology in Stockholm. (*Photo courtesy of Harriet Woodcock.*)