# PART II

# ROTATION EFFECTS

# 4

# *Geostrophic Flows and Vorticity Dynamics*

**Summary:** This chapter treats homogeneous flows with small Rossby and Ekman numbers. The tendency of such flows to display vertical rigidity is demonstrated. Then, the concept of potential vorticity is introduced.

## *4-1 HOMOGENEOUS GEOSTROPHIC FLOWS*

Let us consider rapidly rotating fluids by restricting our attention to situations where the Coriolis acceleration by far dominates the various acceleration terms. Let us further consider homogeneous fluids and ignore frictional effects. Mathematically, we write

$$Ro_T \ll 1, \qquad Ro \ll 1, \qquad Ek \ll 1, \tag{4-1}$$

together with $\rho = 0$ (no density variation). The lowest-order equations governing such homogeneous, frictionless, rapidly rotating fluids are the following simplified forms of equations of motion, (3-25) through (3-28):

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{4-2}$$

$$+ fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \tag{4-3}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \tag{4-4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4-5}$$

where $f$ is the Coriolis parameter. This reduced set of equations has a number of surprising properties.

If we take the vertical derivative of the first equation, (4-2), we obtain, successively,

$$-f\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right) = -\frac{1}{\rho_0}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial z}\right) = 0,$$

where the right-hand side vanishes because of (4-4). The other horizontal momentum equation, (4-3), succumbs to the same fate, bringing us to conclude that the vertical derivative of the horizontal velocity must be identically zero:

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0. \tag{4-6}$$

This result is known as the Taylor–Proudman theorem (Proudman, 19 3). Physically, this means that the horizontal velocity field has no vertical shear and that all particles on the same vertical move in concert. Such vertical rigidity is a fundamental property of rotating homogeneous fluids.

Next, let us solve the momentum equations in terms of the velocity components, a trivial task; we get

$$u = \frac{-1}{\rho_0 f}\frac{\partial p}{\partial y}, \qquad v = \frac{+1}{\rho_0 f}\frac{\partial p}{\partial x}, \tag{4-7}$$

with the corollary that the vector velocity $(u, v)$ is perpendicular to the vector $(\partial p/\partial x, \partial p/\partial y)$. Since the latter vector is none other than the pressure gradient, we conclude that the flow is not down-gradient but rather across-gradient. The fluid particles are not cascading from high to low pressures, as in a nonrotating viscous fluid but, instead, are navigating along lines of constant pressures, called *isobars* (Figure 4-1). (The flow is said to be isobaric, and isobars are streamlines.) This implies that no pressure work is performed either on the fluid or by the fluid. Hence, once initiated, the flow can persist without a continuous energy source.

Such a flow field, where a balance is struck between the Coriolis and pressure forces, is called *geostrophic* (from the Greek, γη = Earth and στροφη = turning). The property is called *geostrophy*. Hence, by definition, all geostrophic flows are isobaric.

A remaining question concerns the direction of flow along the pressure lines. A quick examination of the signs in expressions (4-7) reveals that, where $f$ is positive (Northern Hemisphere, counterclockwise ambient rotation), the currents flow with the

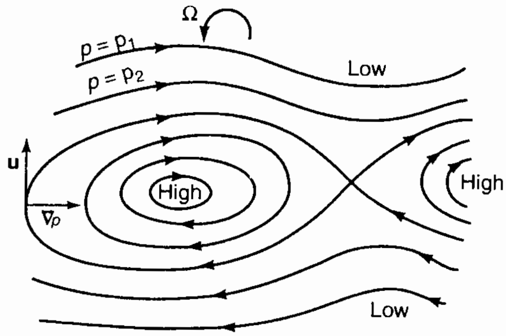


**Figure 4-1** Example of geostrophic flow. The velocity vector is everywhere parallel to the lines of equal pressure. Thus, pressure contours are streamlines. In the Northern Hemisphere (as pictured here), the fluid circulates with the high pressure on its right. The opposite holds for the Southern Hemisphere.

high pressures on their right. Where $f$ is negative (Southern Hemisphere, clockwise ambient rotation), they flow with the high pressures on their left. Figure 4-2 provides a meteorological example from the Northern Hemisphere.

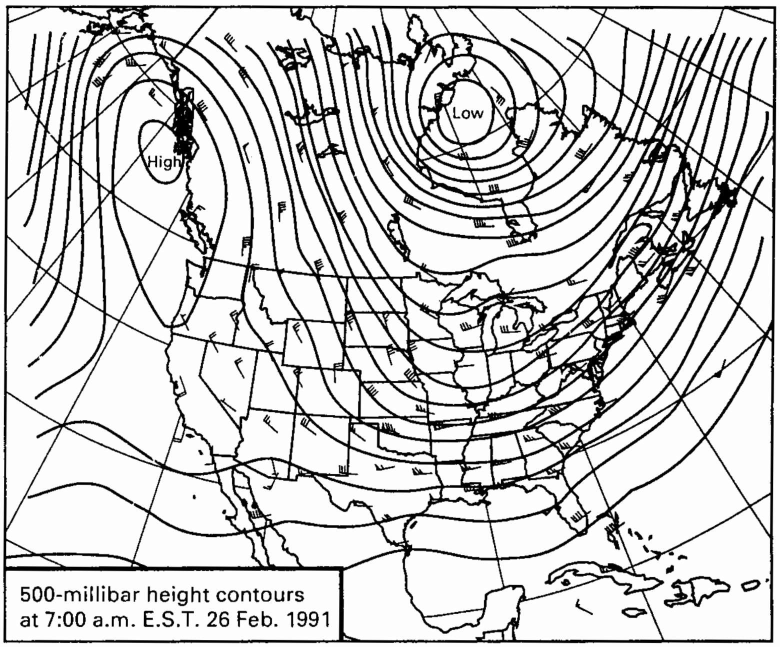


**Figure 4-2** A meteorological example showing the high degree of parallelism between wind velocities and pressure contours (isobars), indicative of approximate geostrophic balance. The solid lines are height contours of a given pressure (500 mb) and not pressure contours at a given height. However, because atmospheric pressure variations are large vertically and weak horizontally, the two sets of contours are nearly identical. According to meteorological convention, wind vectors are depicted by arrows with flags and barbs; on each arrow tail, a flag indicates a speed of 50 knots, a barb 10 knots, and a half-barb 5 knots (1 knot = 1 nautical mile per hour = 0.5144 m/s); the wind is directed toward the bare end of the arrow. (Chart prepared by National Weather Service, Department of Commerce, Washington, D.C.)

If the flow field extends over a meridional span that is not too wide, the variation of the Coriolis parameter with latitude is negligible, and $f$ can be taken as a constant. The frame of reference is then called the *f-plane*. In this case, the horizontal divergence of the geostrophic flow vanishes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial}{\partial x}\left(\frac{1}{\rho_0 f}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho_0 f}\frac{\partial p}{\partial x}\right) = 0. \qquad (4\text{-}8)$$

Hence, geostrophic flows are naturally nondivergent on the $f$-plane. This leaves no room for vertical convergence or divergence, as the continuity equation (4-5) implies:

$$\frac{\partial w}{\partial z} = 0. \qquad (4\text{-}9)$$

A corollary is that the vertical velocity, too, is independent of depth. If the fluid is limited in the vertical by a flat bottom (horizontal ground or sea for the atmosphere) or by a flat surface (sea surface for the ocean), this vertical velocity must simply vanish, and the flow is strictly two-dimensional.

## 4-2 HOMOGENEOUS GEOSTROPHIC FLOWS OVER AN IRREGULAR BOTTOM

Let us still consider a rapidly rotating fluid, so that the flow is geostrophic, but now over an irregular bottom. We neglect the eventual surface displacements, assuming that they remain modest in comparison with the bottom irregularities (Figure 4-3). An example would be the flow in a shallow sea (homogeneous waters) with depth ranging from 20 to 50 m and under surfaces waves of a few tens of centimeters high.

If the flow were to climb up or down the bottom, it would undergo a vertical velocity proportional to the slope:

$$w = u\,\frac{\partial}{\partial x}(H-h) + v\,\frac{\partial}{\partial y}(H-h) = -u\,\frac{\partial h}{\partial x} - v\,\frac{\partial h}{\partial y}, \qquad (4\text{-}10)$$

where $h$ is the fluid depth measured from the surface and $H$ is a constant reference depth

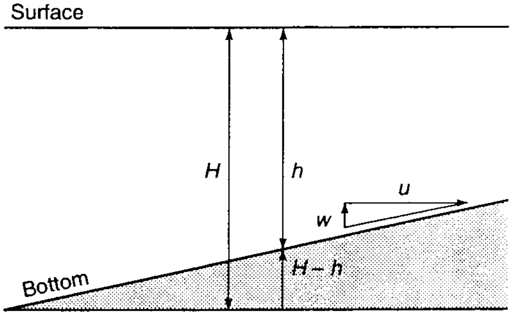


**Figure 4-3** Schematic view of a flow over a sloping bottom. A vertical velocity must accompany flow across isobaths.

(so that $H - h$ is the bottom elevation above the reference level). The analysis of the previous section implies that the vertical velocity is constant across the entire depth of the fluid. Since it must be zero at the surface, it must be so at the bottom as well; that is,

$$u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} = 0, \qquad (4\text{-}11)$$

and the flow is prevented from climbing up or down the bottom slope. This property has profound implications. In particular, if the topography consists of an isolated bump (or dip) in an otherwise flat bottom, the fluid on the flat bottom cannot rise onto the bump, even partially, and must instead go around it. Because of the vertical rigidity of the flow, the fluid particles at all levels—including levels above the bump elevation—must likewise go around. Similarly, the fluid over the bump cannot leave the bump and must remain there. Such permanent tubes of fluids above bumps or cavities are called *Taylor columns*.

In regions of flat bottom, a geostrophic flow can assume arbitrary patterns, and the actual pattern reflects the initial conditions. But, over a bottom where the slope is nonzero almost everywhere (Figure 4-4), the geostrophic flow has no choice but to follow the depth contours (called *isobaths*). Pressure contours are then aligned with topographic contours, and isobars are isobaths. These lines are sometimes also called *geostrophic contours*. Open isobaths that run from boundary to boundary cannot support any flow, otherwise fluid would be required to enter or leave through lateral boundaries. The flow is simply blocked there. Therefore, free geostrophic flow can occur only along closed isobaths.

The preceding conclusions hold true as long as the upper boundary is horizontal. If this is not the case, it can then be shown that geostrophic flows are constrained to be directed along lines of constant fluid depth. (See Problem 4-3.) Thus, the fluid is allowed to move up and down, but only as long as it is not being vertically squeezed or stretched. This property is a direct consequence of the inability of geostrophic flows to undergo any two-dimensional divergence.

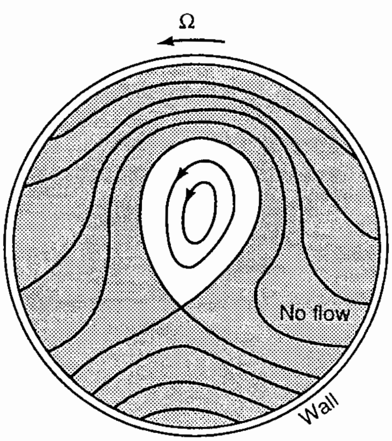


**Figure 4-4** Geostrophic flow in a closed domain and over irregular topography. Solid lines are isobaths (contours of equal depth). Flow is permitted only along closed isobaths.

### 4-3 GENERALIZATION TO NONGEOSTROPHIC FLOWS

Let us no longer suppose that the fluid is rapidly rotating (i.e., the Coriolis acceleration no longer dwarfs other acceleration terms) but still suppose that it is homogeneous and frictionless. The equations are now augmented to include the relative-acceleration terms:

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\,\frac{\partial p}{\partial x} \tag{4-12a}$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\,\frac{\partial p}{\partial y}. \tag{4-12b}$$

The pressure still obeys (4-4), and the continuity equation, (4-5), has not changed.

If the horizontal flow field is initially independent of depth, it will remain so at all future times. Indeed, the nonlinear advection terms and the Coriolis terms are initially $z$-independent, and the pressure terms are also $z$-independent by virtue of (4-4). Thus, $\partial u/\partial t$ and $\partial v/\partial t$ must be $z$-independent, which implies that $u$ and $v$ tend not to become depth-varying and thus remain $z$-independent at all subsequent times. Let us restrict our attention to such flows, which in the jargon of geophysical fluid dynamics are called *barotropic*. Equations (4-12a) and (4-12b) then reduce to

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} - fv = -\frac{1}{\rho_0}\,\frac{\partial p}{\partial x} \tag{4-13a}$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho_0}\,\frac{\partial p}{\partial y}. \tag{4-13b}$$

Although the flow has no vertical structure, the similarity to geostrophic flow ends there. In particular, the flow is not required to be aligned with the isobars, nor is it devoid of vertical velocity. To determine the vertical velocity, we turn to the continuity equation, (4-5),

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

in which we note that the first two terms are independent of $z$ but do not necessarily add up to zero. A vertical velocity varying linearly with depth can exist, enabling the flow to support two-dimensional divergence and thus allowing a flow across isobaths.

An integration of the preceding equation over the entire fluid depth yields

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \int_b^{b+h} dz + [w]_b^{b+h} = 0, \tag{4-14}$$

where $b$ is the bottom elevation above a reference level and $h$ is the local and instantaneous fluid depth (Figure 4-5). Because fluid particles on the surface cannot leave the surface and particles on the bottom cannot leave the bottom, the vertical velocities at those levels are given by

$$w(z = b + h) = \frac{\partial}{\partial t}(b + h) + u \frac{\partial}{\partial x}(b + h) + v \frac{\partial}{\partial y}(b + h)$$

$$w(z = b) = u \frac{\partial b}{\partial x} + v \frac{\partial b}{\partial y}.$$

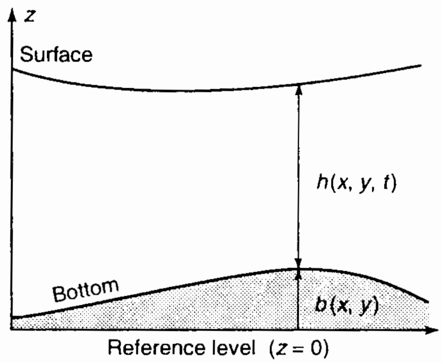


**Figure 4-5** Schematic diagram of an unsteady flow of a homogeneous fluid over an irregular bottom and the corresponding notation.

Equation (4-14) then becomes

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0, \tag{4-15}$$

which supersedes (4-5) and eliminates the vertical velocity from the formalism.

Finally, since the fluid is homogeneous, the dynamic pressure, $p$, is independent of depth. In the absence of a pressure variation above the fluid surface (e.g., uniform atmospheric pressure over the ocean), the dynamic pressure is

$$p = \rho_0 g\,(h + b), \tag{4-16}$$

where $g$ is the gravitational acceleration. [To verify this, calculate the total hydrostatic pressure and subtract the term linear in $z$, the $p_0$ field of (3-15).] With $p$ replaced by the preceding expression, equations (4-13a), (4-13b) and (4-15) form a 3-by-3 system for the variables $u$, $v$, and $h$. The vertical variable no longer appears, and the independent variables are $x$, $y$, and $t$.

If the bottom is flat, the system becomes

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - fv = -g \frac{\partial h}{\partial x}, \tag{4-17}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + fu = -g \frac{\partial h}{\partial y}, \tag{4-18}$$

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) = 0. \tag{4-19}$$

Although this system of equations is applied as frequently to the atmosphere as to the ocean, it bears the name *shallow-water model.*

## 4-4 VORTICITY DYNAMICS

In the study of geostrophic flows (Section 4-1), it was noted that the pressure terms cancel in the expression of the two-dimensional divergence. Let us now repeat this operation while keeping the added acceleration terms by subtracting the $y$-derivative of (4-13a) from the $x$-derivative of (4-13b). After some algebra, the result can be cast as follows:

$$\frac{d}{dt}\left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\left(f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0, \qquad (4\text{-}20)$$

where the material time derivative is defined, as previously, by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\,\frac{\partial}{\partial x} + v\,\frac{\partial}{\partial y}.$$

In the derivation, care was taken to allow for the possibility of a variable Coriolis parameter (which on a sphere varies with latitude and thus with position). The grouping

$$f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = f + \zeta \qquad (4\text{-}21)$$

is interpreted as the sum of the ambient vorticity ($f$) and the relative vorticity ($\zeta = \partial v/\partial x - \partial u/\partial y$). To be precise, the vorticity is a vector, but since the horizontal flow field has no depth-dependence, there is no vertical shear and no eddies with horizontal axes. The vorticity vector is strictly vertical, and the preceding expression merely shows that vertical component.

Similarly, the continuity equation, (4-15), can be expanded into

$$\frac{d}{dt}\,h + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) h = 0. \qquad (4\text{-}22)$$

Then a third and final equation of this type can be written for the cross section of an infinitesimal fluid column. Consider a fluid column of cross-section $ds$: As it moves with the flow, it is translated, rotated, strained, sheared, and compressed or expanded. So, its cross-section changes. The equation governing those changes is

$$\frac{d}{dt}\,ds = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) ds. \qquad (4\text{-}23)$$

Physically, a horizontal divergence causes an increase in cross-section and a convergence a decrease in cross-section. Combining (4-22) and (4-23), we obtain

$$\frac{d}{dt}\,(h\;ds) = 0, \qquad (4\text{-}24)$$

which simply states that the parcel's volume is conserved in time. If the parcel is squeezed vertically, it stretches horizontally, and vice versa (Figure 4-6).

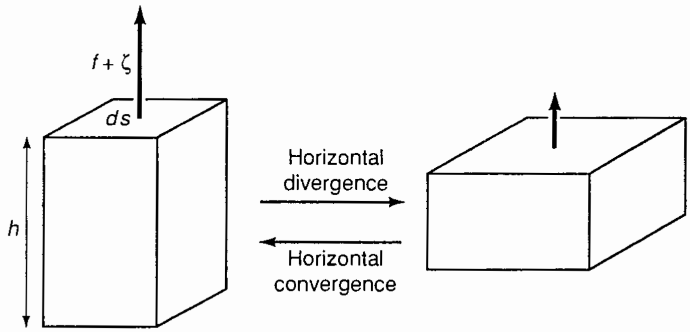


**Figure 4-6** Conservation of volume and circulation of a fluid parcel undergoing squeezing or stretching. The products $h\ ds$ and $(f + \zeta)\ ds$ are conserved during the transformation. As a corollary, the ratio $(f + \zeta)/h$, called potential vorticity, is also conserved.

A similar combination of (4-20) and (4-23) yields

$$\frac{d}{dt}\left[(f + \zeta)\ ds\right] = 0 \tag{4-25}$$

and implies that the product $(f + \zeta)\ ds$ is conserved by the fluid parcel. This product can be interpreted as the vorticity flux (vorticity integrated over the cross-section) and is therefore the circulation of the parcel. Equation (4-25) is the particular expression for rotating, two-dimensional flows of Kelvin's theorem, which guarantees conservation of circulation in inviscid fluids (Batchelor, 1967).

This conservation principle is akin to that of angular momentum for an isolated system. The best example is that of a ballerina spinning on her toes; with her arms stretched out, she spins slowly, but with her arms close to her body, she spins more rapidly. In homogeneous geophysical flows, when a parcel of fluid is squeezed laterally ($ds$ decreasing), its vorticity must increase ($f + \zeta$ increasing) to conserve circulation.

Now, if both circulation and volume are conserved, so is their ratio. This ratio is particularly helpful, for it eliminates the parcel's cross-section and thus depends only on local variables of the flow field:

$$\frac{d}{dt}\left(\frac{f + \zeta}{h}\right) = 0, \tag{4-26}$$

where

$$q = \frac{f + \zeta}{h} = \frac{f + \partial v/\partial x - \partial u/\partial y}{h} \tag{4-27}$$

is called the *potential vorticity*. The preceding analysis interprets potential vorticity as circulation per volume. This quantity, as will be shown on a numerous occasions in this book, plays a fundamental role in geophysical flows. Note that equation (4-26) could have been derived directly from (4-20) and (4-22) without recourse to the introduction of the variable $ds$.

Let us now go full circle and return to rapidly rotating flows, those in which the Coriolis force dominates. In this case, the Rossby number is much less than unity ($Ro = U/\Omega L \ll 1$), which implies that the relative vorticity ($\zeta = \partial v/\partial x - \partial u/\partial y$, scaling like $U/L$) is negligible in front of the ambient vorticity ($f$, scaling like $\Omega$). The potential vorticity reduces to

$$q = \frac{f}{h} \tag{4-28}$$

which, if $f$ is constant—such as in a rotating laboratory tank or for geophysical patterns of modest meridonal extent—implies that each fluid column must conserve its height $h$. In particular, if the upper boundary is horizontal, fluid parcels must follow isobaths.

## PROBLEMS

**4-1.** A laboratory experiment is conducted in a cylindrical tank 20 cm in diameter, filled with homogeneous (15 cm deep at the center) water and rotating at 30 rpm. A steady flow field with maximum velocities of 1 cm/s is generated by a source-sink device. The water viscosity is $10^{-6}$ m$^2$/s. Verify that this flow field meets the conditions of geostrophy.

**4-2.** (Generalization of the Taylor–Proudman theorem) By reinstating the $f_*$-terms of equations (3-13), (3-14), and (3-17) into (4-2) through (4-4), show that motions in fluids rotating rapidly around an axis not parallel to gravity exhibit columnar behavior in the direction of the axis of rotation.

**4-3.** Demonstrate the assertion made at the end of Section 4-2, namely, that geostrophic flows between irregular bottom and top boundaries are constrained to be directed along lines of constant fluid depth.

**4-4.** Establish equation (4-23) for the evolution of a parcel's horizontal cross-section from first principles.

**4-5.** In a fluid of depth $H$ rapidly rotating at the rate $\Omega$ (Figure 4-7), there exists a uniform flow $U$. Along the bottom (fixed), there is an obstacle of height $H'$ ($< H/2$), around which the flow is locally deflected, leaving a quiescent Taylor column. A rigid lid, translating in the direction of the flow at the speed $2U$, has a protrusion identical to the bottom obstacle, also locally deflecting the otherwise uniform flow and entraining another quiescent Taylor column. The two obstacles are aligned with the motion axis so that there will be a time when both are superimposed. Assuming that the fluid is homogeneous and frictionless, what do you think will happen to the Taylor columns?

**4-6.** As depicted in Figure 4-8, a vertically uniform but laterally sheared coastal current must climb a bottom escarpment. Assuming that the jet velocity still vanishes offshore, determine

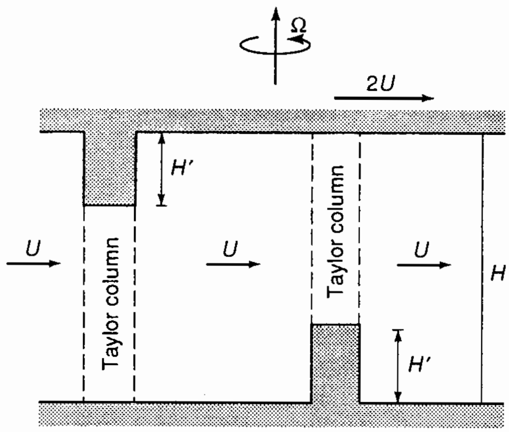


Figure 4-7 Schematic view of a hypothetical system, as described in Problem 4-5.

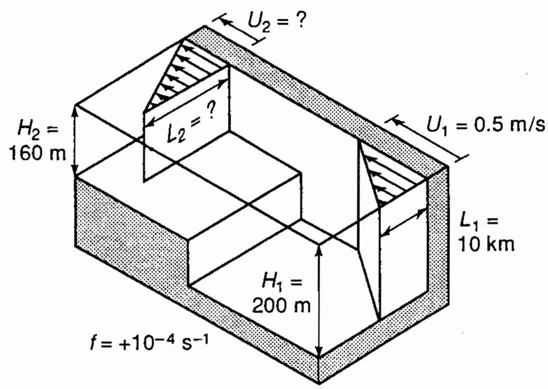


Figure 4-8 A sheared coastal jet negotiating a bottom escarpment (Problem 4-6).

the velocity profile and the width of the jet downstream of the escarpment. What would happen if the downstream depth were only 100 m?

**4-7.** What are the differences in dynamic pressure across the coastal jet of Problem 4-6 upstream and downstream of the escarpment? Take $H_2 = 160$ m and $\rho_0 = 1022$ kg/m$^3$.

**4-8.** In Utopia, a narrow 200-m deep channel empties in a broad bay of varying bottom topography (Figure 4-9). Trace the path to the sea and the velocity profile of the channel outflow. Take $f = 10^{-4}$ s$^{-1}$. (Solve only for straight stretches of the flow and not for corners.)

**4-9.** A steady ocean current of uniform potential vorticity $q = 5 \times 10^{-7}$ m$^{-1}\cdot$s$^{-1}$ and volume flux $T = 4 \times 10^5$ m$^3$/s flows along isobaths of a uniformly sloping bottom (bottom slope $s = 1$ m/km). Show that the velocity profile across the current is parabolic. What are the width of the current and the depth of the location of maximum velocity? (Take $f = 7 \times 10^{-5}$ s$^{-1}$.)

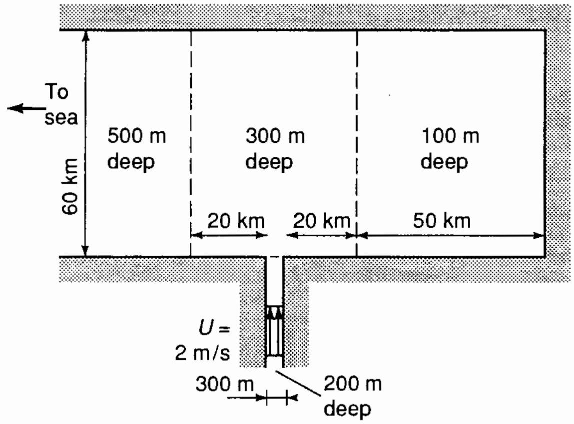


**Figure 4-9** Geometry of the idealized bay and channel mentioned in Problem 4-8.

## SUGGESTED LABORATORY DEMONSTRATION

### *Equipment*

A cylindrical rotating tank with flat bottom and smooth sides, a cylindrical obstacle.

### *Experiment*

Place the cylindrical obstacle at the bottom of the tank, somewhat off center. Fill the tank so that the obstacle does not occupy more than one quarter of the depth. Bring the tank to solid-body rotation. Inject some dye in the fluid away from the obstacle and, if possible, some dye of a different color above the obstacle. Allow time for vertical sheets to form. Then, reduce the rotation rate slightly so that the fluid, still rotating at the old rate for a while, flows with respect to the tank, which is rotating at the new rate. Note how, upon encountering the obstacle, the vertical sheets deflect and go around it. Also note how the fluid above the obstacle remains above the obstacle, forming a Taylor column.

## Geoffrey Ingram Taylor

**1886 – 1975**

Considered one of the great physicists of this century, Sir Geoffrey Taylor contributed enormously to our understanding of fluid dynamics. Although he did not envision the birth and development of geophysical fluid dynamics, his research on rotating fluids laid the foundation for the discipline. His numerous contributions also include seminal work on turbulence, aeronautics, and solid mechanics. With a staff consisting of a single assistant-engineer, he maintained a very modest laboratory, constantly preferring to undertake entirely new problems and to work alone. (*Photo courtesy of Cambridge University Press.*)