6

Linear Barotropic Waves

Summary: The aim of this chapter is to describe an assortment of waves that can be supported by an inviscid, homogeneous fluid in rotation.

6-1 LINEAR WAVE DYNAMICS

Chiefly because linear equations are most amenable to methods of solution, it is wise to gain insight into geophysical fluid dynamics by elucidating the possible linear processes and investigating their properties before exploring more intricate, nonlinear dynamics. The governing equations (Section 3-5) are essentially nonlinear; consequently, their linearization can proceed only by imposing restrictions on the flows under consideration.

The Coriolis acceleration terms present in the momentum equations [(3-25) and (3-26)] are, by nature, linear and need not be subjected to any approximation. This situation is extremely fortunate because these are the central terms of geophysical fluid dynamics. In contrast, the so-called advective terms (or convective terms) are quadratic and undesirable at this moment. Hence, our considerations will be restricted to low-Rossby-number situations:

$$Ro = \frac{U}{\Omega L} \ll 1. \tag{6-1}$$

This is usually accomplished by restricting the attention to relatively weak flows (small U), large scales (large L), or, in the laboratory, fast rotation (large Ω). The terms expressing the local time rate of change of the velocity ($\partial u/\partial t$ and $\partial v/\partial t$) are linear and are retained here in order to permit the investigation of unsteady flows. Thus, the temporal Rossby number is taken as

$$Ro_T = \frac{1}{\Omega T} \sim 1. \tag{6-2}$$

Contrasting conditions (6-1) and (6-2), we conclude that we are about to consider slow flow fields that evolve relatively fast. Aren't we asking for the impossible? Not at all, for rapidly moving disturbances do not necessarily require large velocities. In other words, information may travel faster than material particles, and when this is the case, the flow takes the aspect of a wave field. A typical example is the spreading of concentric ripples on the surface of a pond after the throwing of a stone; energy radiates but there is no appreciable water movement across the pond. In keeping with the foregoing quantities, a scale for the wave speed can be defined as the velocity of a signal covering the distance L of the flow during the nominal evolution time T, and, by virtue of restrictions (6-1) and (6-2), it can be compared to the flow velocity:

$$C = \frac{L}{T} \sim \Omega L \gg U. \tag{6-3}$$

Thus, our present objective is to consider wave phenomena.

To shed the best possible light on the mechanisms of the basic wave processes typical in geophysical flows, we will further restrict our attention to homogeneous and inviscid flows, for which the shallow-water model (Section 4-3) is adequate. With all the preceding restrictions, the horizontal momentum equations, (4-17) and (4-18), reduce to

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{6-4}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}, \tag{6-5}$$

where f is the Coriolis parameter, g is the gravitational acceleration, u and v are the velocity components in the x- and y-directions, respectively, and η is the surface displacement obtained from the total fluid depth h by subtraction of the mean fluid depth H (i.e., $\eta = h - H$). The independent variables are x, y, and t; the vertical coordinate is absent, for the flow is depth-invariant (Chapter 4).

In terms of surface displacement, η, the continuity equation (4-19) can be expanded in several groups of terms:

$$\frac{\partial \eta}{\partial t} + \left(u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} \right) + H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \eta \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0$$

if the mean depth H is constant (flat bottom). Introducing the scale ΔH for the vertical displacement η of the surface, we note that the four groups of terms in the above equation are, sequentially, on the order of

$$\frac{\Delta H}{T}, \quad \frac{U \Delta H}{L}, \quad \frac{UH}{L}, \quad \frac{U \Delta H}{L}.$$

According to (6-3), L/T is much larger than U, and the second and fourth groups of terms are to be neglected compared with the first term, leaving us with the linearized equation

$$\frac{\partial \eta}{\partial t} + H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0, \tag{6-6}$$

the balance of which requires $\Delta H/T$ on the order of UH/L or, again by virtue of (6-3),

$$\Delta H \ll H.$$

We are thus restricted to waves of small amplitudes.

The system of equations (6-4) through (6-6) governs the linear wave dynamics of inviscid, homogeneous fluids under rotation. For the sake of simple notation, we will perform the mathematical derivations only for positive values of the Coriolis parameter f and then state the conclusions for both positive and negative values of f. The derivations with negative values of f are left as exercises. Before proceeding with the separate studies of geophysical fluid waves, the student or reader not familiar with the concepts of phase speed, wave-number vector, dispersion relation, and group velocity is directed to Appendix A. A comprehensive account of geophysical waves can be found in the book by LeBlond and Mysak (1978).

6-2 THE KELVIN WAVE

The Kelvin wave is a traveling disturbance that requires the support of a lateral boundary. Therefore, it most often occurs in the ocean where it can travel along coastlines. For convenience, we use oceanic terminology such as coast and offshore.

As a simple model here, consider a semi-infinite layer of fluid bounded below by a horizontal bottom, above by a free surface, and on one side (say, the y-axis) by a vertical wall (Figure 6-1). Along this wall ($x = 0$, the coast), the normal velocity must vanish ($u = 0$), but the absence of viscosity allows a nonzero tangential velocity.

As he recounted in his presentation to the Royal Society of Edinburg in 1879, Sir William Thomson (later to become Lord Kelvin) thought that the vanishing of the velocity component normal to the wall suggested the possibility that it be zero everywhere. So, let us state, in anticipation,

$$u = 0$$

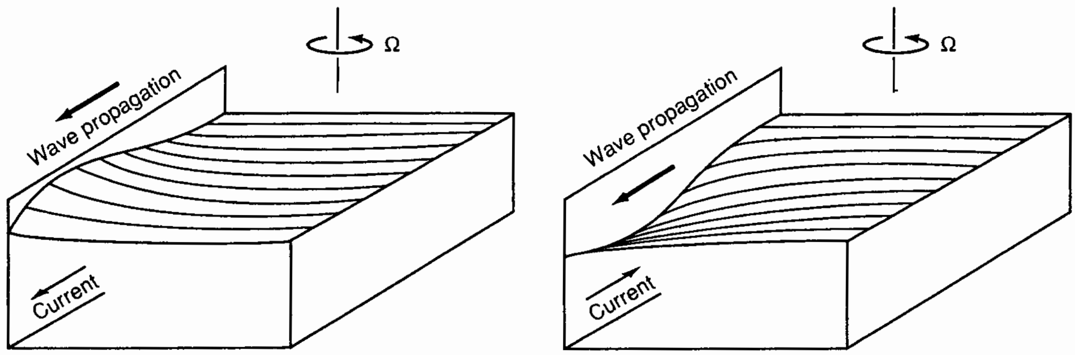


Figure 6-1 Upwelling and downwelling Kelvin waves. In the Northern Hemisphere, both waves travel with the coast on their right, but the accompanying currents differ.

and investigate the consequences. Although equation (6-4) contains a remaining derivative with respect to x, equations (6-5) and (6-6) contain only derivatives with respect to y and time. Elimination of the surface displacement leads to a single equation for the longshore velocity:

$$\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial y^2}, \tag{6-7}$$

where

$$c = \sqrt{gH} \tag{6-8}$$

is identified as the speed of surface gravity waves in nonrotating shallow waters. The preceding equation governs the propagation of one-dimensional nondispersive waves and possesses the general solution

$$v = V_1(x, y + ct) + V_2(x, y - ct), \tag{6-9}$$

which consists of two waves, one traveling toward decreasing y and the other in the opposite direction. Returning to either (6-5) or (6-6) where u is set to zero, we easily determine the surface displacement:

$$\eta = -\sqrt{\frac{H}{g}}\; V_1(x, y + ct) + \sqrt{\frac{H}{g}}\; V_2(x, y - ct).$$

(Any additive constant can be eliminated by a proper redefinition of the mean depth H.) The structure of these functions V_1 and V_2 is then determined by the use of the remaining equation, (6-4):

$$\frac{\partial V_1}{\partial x} = -\frac{f}{\sqrt{gH}}\, V_1, \qquad \frac{\partial V_2}{\partial x} = +\frac{f}{\sqrt{gH}}\, V_2$$

or

$$V_1 = V_{10}(y + ct)\ e^{-x/R}, \qquad V_2 = V_{20}(y - ct)\ e^{+x/R},$$

where the length R, defined as

$$R = \frac{\sqrt{gH}}{f} = \frac{c}{f}, \tag{6-10}$$

combines all three constants of the problem. Within a numerical factor, it is the distance covered by a wave, such as the present one, traveling at the speed c during one inertial period ($2\pi/f$). For reasons that will become apparent later, this quantity is called the *Rossby radius of deformation*, or, more simply, the radius of deformation.

Of the two independent solutions, the second increases exponentially with distance from shore and is declared physically unfit. This leaves as the most general solution:

$$u = 0 \tag{6-11a}$$

$$v = \sqrt{gH}\ F(y + ct)\ e^{-x/R} \tag{6-11b}$$

$$\eta = -H\ F(y + ct)\ e^{-x/R}, \tag{6-11c}$$

where F is an arbitrary function of its variable.

Because of the exponential decay away from the boundary, the Kelvin wave is said to be trapped. Without the boundary, it is unbounded at large distances and thus cannot exist; the length R is a measure of the trapping distance. In the longshore direction, the wave travels without distortion at the speed of surface gravity waves. In the Northern Hemisphere ($f > 0$, as in the preceding analysis), the wave travels with the coast on its right; in the Southern Hemisphere, with the coast on its left. Note that, although the direction of wave propagation is unique, the sign of the longshore velocity is arbitrary: An upwelling wave (i.e., a surface bulge with $\eta > 0$) has a current flowing in the direction of the wave, whereas a downwelling wave (i.e., a surface trough with $\eta < 0$) is accompanied by a current flowing in the direction opposite to that of the wave (Figure 6-1).

In the limit of no rotation ($f \to 0$), the trapping distance increases without bound and the wave reduces to a simple gravity wave with crests and troughs oriented perpendicularly to the coast.

Surface Kelvin waves (as described previously, and to be distinguished from internal Kelvin waves, which require a stratification, see the end of Chapter 12) are generated by the ocean tides and by local wind effects in coastal areas. For example, a storm off the northeast coast of Great Britain can send a Kelvin wave that follows the shores of the North Sea in a counterclockwise direction and eventually reaches the west coast of Norway. Traveling in approximately 40 m of water and over a distance of 2200 km, it accomplishes its journey in about 31 h.

The decay of the Kelvin-wave amplitude away from the coast is clearly manifested in the English Channel. The North Atlantic tide enters the Channel from the west and travels eastward toward the North Sea (Figure 6-2). Being essentially a surface wave in a rotating fluid bounded by a coast, the tide assumes the character of a Kelvin wave and propagates while leaning against a coast on its right, namely, France. This explains

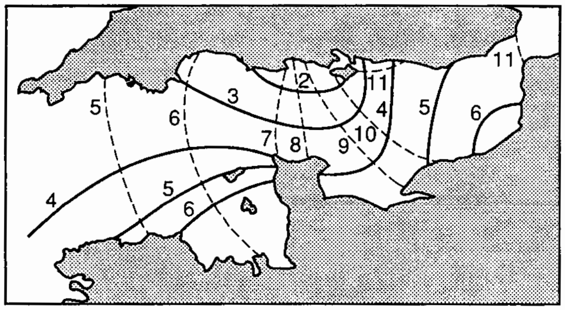


Figure 6-2 Cotidal lines (dashed) with time in lunar hours for the tide in the English channel showing the eastward progression of the tide from the North Atlantic Ocean. Lines of equal tidal range (solid, with values in meters) reveal larger amplitudes along the French coast, namely to the right of the wave progression in accordance with Kelvin waves. (From Proudman, 1953, as adapted by Gill, 1982.)

why tides are noticeably higher along the French coast than along the British coast a few tens of kilometers across (Figure 6-2).

6-3 INERTIA-GRAVITY WAVES (POINCARÉ WAVES)

Let us now do away with the lateral boundary and relax the stipulation $u = 0$. The system of equations (6-4) through (6-6) is kept in its entirety. With f constant and in the presence of a flat bottom, all coefficients are constant, and a Fourier-mode solution can be sought. With u, v, and η taken as constant factors times the function

$$e^{i(lx + my - \omega t)},$$

where l and m are the wave numbers in the x- and y-directions, respectively, and ω is a frequency, the system of equations becomes algebraic:

$$-i\omega u - fv = -igl\eta, \tag{6-12a}$$

$$-i\omega v + fu = -igm\eta, \tag{6-12b}$$

$$-i\omega\eta + H(ilu + imv) = 0. \tag{6-12c}$$

This system admits the trivial solution $u = v = \eta = 0$ unless its determinant vanishes. Thus waves occur only when the following condition is met:

$$\omega\,[\omega^2 - f^2 - gH\,(l^2 + m^2)] = 0. \tag{6-13}$$

This condition, called the *dispersion relation*, provides the wave frequency in terms of the wave-number magnitude $k = (l^2 + m^2)^{1/2}$ and the constants of the problem. The first root, $\omega = 0$, corresponds to a steady state. Returning to (6-4) through (6-6) with the time derivatives set to zero, we recognize the equations governing the geostrophic flow described in Section 4-1. In other words, geostrophic flows can be interpreted as arrested waves. The remaining two roots,

$$\omega = \sqrt{f^2 + gHk^2} \tag{6-14}$$

and its opposite, correspond to bona fide traveling waves, called *Poincaré waves*. In the limit of no rotation ($f = 0$), the frequency is $\omega = k\sqrt{gH}$ and the phase speed is $c = \omega/k = \sqrt{gH}$. The waves become classical gravity waves. The same limit also occurs at large wave numbers [$k^2 \gg f^2/gH$, i.e., wavelengths much shorter than the deformation radius defined in (6-10)]. This is not too surprising, since such waves are too short to feel the rotation of the earth. At the opposite extreme of low wave numbers ($k^2 \ll f^2/gH$, i.e., wavelengths much longer than the deformation radius), the rotation effect dominates, yielding $\omega \simeq f$. At this limit, the flow pattern is virtually laterally uniform, and all fluid particles move in unison, each describing a circular inertial oscillation, as described in Section 2-4. For intermediate wave numbers, the frequency (Figure 6-3) is always greater than f, and the waves are said to be *superinertial*. Since Poincaré waves exhibit a mixed behavior between gravity waves and inertial oscillations, they are also called *inertia-gravity waves*.

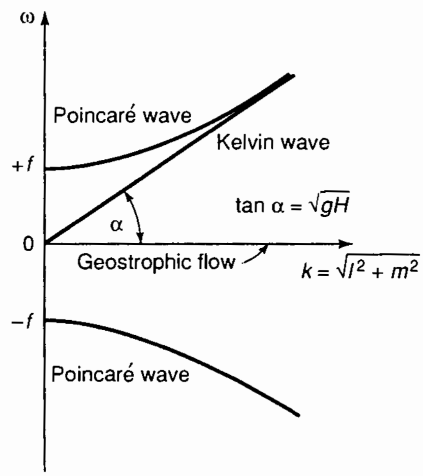


Figure 6-3 Recapitulation of the dispersion relation of Kelvin and Poincaré waves on the f-plane and on a flat bottom.

Because the phase speed $c = \omega/k$ depends on the wave number, wave components of different wavelengths travel at different speeds, and the wave is said to be *dispersive*. This is in contrast with the nondispersive Kelvin wave, whose signal travels without distortion, irrespective of its profile. See Appendix A for additional information on these notions.

6-4 PLANETARY WAVES (ROSSBY WAVES)

Kelvin and Poincaré waves are relatively fast waves, and we may wonder whether rotating, homogeneous fluids could not support another breed of slower waves. Could it be, for example, that the steady geostrophic flows, those corresponding to the zero-frequency solution found in the preceding section, develop a slow evolution (frequency slightly above zero) when the system is slightly modified? The answer is yes, and forming one class are the planetary waves, in which the time evolution is prompted by the weak but important *planetary effect.*

As we may recall from Section 2-6, on a spherical earth (or planet or star, in general), the Coriolis parameter, f, is proportional to the rotation rate, Ω, times the sine of the latitude, φ:

$$f = 2\Omega \sin \varphi.$$

Large wave formations such as alternating cyclones and anticyclones contributing to our daily weather and, to a lesser extent, the Gulf Stream meanders span several degrees of latitude; for them, it is necessary to consider the meridional change in the Coriolis parameter. If the coordinate y is oriented northward and is measured from a reference latitude φ_0 (say, a latitude somewhere in the middle of the wave under consideration), then $\varphi = \varphi_0 + y/a$, where a is the earth's radius (6371 km). Considering y/a as a small departure, the Coriolis parameter can be expanded in a Taylor series:

$$f = 2\Omega \sin \varphi_0 + 2\Omega \frac{y}{a} \cos \varphi_0 + \cdots \tag{6-15}$$

Retaining only the first two terms, we write in traditional notation

$$f = f_0 + \beta_0 y, \tag{6-16}$$

where $f_0 = 2\Omega \sin \varphi_0$ is the reference Coriolis parameter and $\beta_0 = 2(\Omega/a) \cos \varphi_0$ is the *beta parameter.* Typical midlatitude values on Earth are $f_0 = 8 \times 10^{-5} \cdot s^{-1}$ and $\beta_0 = 2 \times 10^{-11}\ m^{-1} \cdot s^{-1}$. The Cartesian framework where the beta term is not retained is called the *f-plane*, and that where it is retained is called the *beta plane.* The next step in order of accuracy is to retain the full spherical geometry (which we will avoid throughout this book). Rigorous justifications of the beta-plane approximation can be found in Veronis (1963, 1981), Pedlosky (1987), and Verkley (1990).

Note that the beta-plane representation is validated at mid latitudes only if the $\beta_0 y$ term is small compared to the leading f_0 term. In terms of the motion's meridional length scale L, this implies

$$\beta = \frac{\beta_0 L}{f_0} \ll 1, \tag{6-17}$$

where the dimensionless ratio can be called the *planetary number.*

The governing equations, having become

$$\frac{\partial u}{\partial t} - (f_0 + \beta_0 y) v = -g \frac{\partial \eta}{\partial x}, \tag{6-18a}$$

$$\frac{\partial v}{\partial t} + (f_0 + \beta_0 y) u = -g \frac{\partial \eta}{\partial y}, \tag{6-18b}$$

$$\frac{\partial \eta}{\partial t} + H \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0, \tag{6-18c}$$

are now mixtures of small and large terms. The larger ones (f_0, g, and H terms) comprise the otherwise steady, f-plane geostrophic dynamics; the smaller ones (time derivatives and β_0 terms) come as perturbations, which, although small, will govern the wave evolution. In first approximation, the large terms dominate, and thus $u \simeq -(g/f_0)\partial\eta/\partial y$ and $v \simeq (g/f_0)\partial\eta/\partial x$. Use of this first approximation in the small terms of (6-18a) and (6-18b) yields

$$-\frac{g}{f_0}\frac{\partial^2\eta}{\partial y\partial t} - f_0 v - \frac{\beta_0 g}{f_0}\, y\,\frac{\partial\eta}{\partial x} = -g\frac{\partial\eta}{\partial x} \tag{6-19a}$$

$$+\frac{g}{f_0}\frac{\partial^2\eta}{\partial x\partial t} + f_0 u - \frac{\beta_0 g}{f_0}\, y\,\frac{\partial\eta}{\partial y} = -g\frac{\partial\eta}{\partial y}. \tag{6-19b}$$

These equations are trivial to solve with respect to u and v:

$$u = -\frac{g}{f_0}\frac{\partial\eta}{\partial y} - \frac{g}{f_0^2}\frac{\partial^2\eta}{\partial x\partial t} + \frac{\beta_0 g}{f_0^2}\, y\,\frac{\partial\eta}{\partial y} \tag{6-20a}$$

$$v = +\frac{g}{f_0}\frac{\partial\eta}{\partial x} - \frac{g}{f_0^2}\frac{\partial^2\eta}{\partial y\partial t} - \frac{\beta_0 g}{f_0^2}\, y\,\frac{\partial\eta}{\partial x}. \tag{6-20b}$$

These last expressions can be interpreted as consisting of the leading and first-correction terms in a regular perturbation series of the velocity field. We identify the first term of each expansion as the geostrophic velocity. By contrast, the next and small terms are called *ageostrophic*.

Final substitution in continuity equation (6-18c) leads to a single equation for the surface displacement:

$$\frac{\partial\eta}{\partial t} - R^2\frac{\partial}{\partial t}\nabla^2\eta - \beta_0 R^2\frac{\partial\eta}{\partial x} = 0, \tag{6-21}$$

where ∇^2 is the two-dimensional Laplace operator and $R = \sqrt{gH}/f_0$ is the deformation radius, defined in (6-10) and now suitably amended to be a constant. Unlike the original set of equations, this last equation has constant coefficients and a solution of the Fourier type, $\cos(lx + my - \omega t)$, can be sought. The dispersion relation follows:

$$\omega = -\beta_0 R^2 \frac{l}{1 + R^2(l^2 + m^2)}, \tag{6-22}$$

providing the frequency ω as a function of the wave-number components l and m. The waves are called *planetary waves* or *Rossby waves*, in honor of Carl-Gustaf Rossby, who first proposed this wave theory to explain the movement of midlatitude weather patterns. We note immediately that if the beta corrections had not been retained ($\beta_0 = 0$), the frequency would have been nil. This is the $\omega = 0$ solution of Section 6-3, which corresponds to a steady geostrophic flow on the f-plane. The absence of the other two roots is explained by our approximation. Indeed, treating the time derivatives as small terms (i.e., having in effect assumed a very small temporal Rossby number, $Ro_T \ll 1$), we have retained only the low frequency, the one much less than f_0.

That the frequency given by (6-22) is indeed small can be verified easily. With L ($\sim 1/l \sim 1/m$) as a measure of the wavelength, two cases can arise: either $L \lesssim R$ or $L \gtrsim R$; the frequency scale is then given, respectively, by

$$\text{Shorter waves:}\quad L \lesssim R, \qquad \omega \sim \beta_0 L \tag{6-23}$$

$$\text{Longer waves:}\quad L \gtrsim R, \qquad \omega \sim \frac{\beta_0 R^2}{L} \lesssim \beta_0 L. \tag{6-24}$$

In either case, our premise in (6-17) that $\beta_0 L$ is much less than f_0 implies that ω is much smaller than f_0 (subinertial wave), as we anticipated.

Let us now explore other properties of planetary waves. First and foremost, the zonal phase speed

$$c_x = \frac{\omega}{l} = \frac{-\beta_0 R^2}{1 + R^2(l^2 + m^2)} \tag{6-25}$$

is always negative, implying a phase propagation to the west (Figure 6-4). The sign of the meridional phase speed $c_y = \omega/m$ is undetermined, since the wave number m may

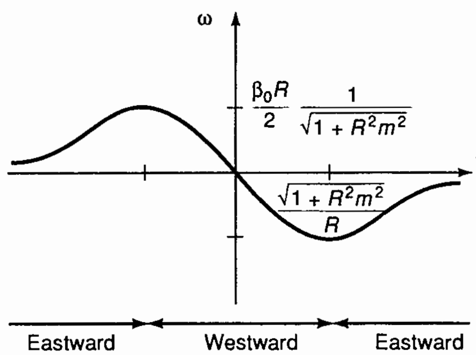


Figure 6-4 Dispersion relation of planetary (Rossby) waves. The frequency ω is plotted against the zonal wave number l at constant meridional wave number m. As the slope of the curve reverses, so does the direction of zonal propagation of energy.

have either sign. Thus, planetary waves can propagate only northwestward, westward, or southwestward. Second, very long waves ($1/l$ and $1/m$ both much larger than R) propagate strictly westward and at the speed

$$c = -\beta_0 R^2, \tag{6-26}$$

which is the maximum wave speed allowed.

Lines of constant frequency, ω, in the (l, m) wave-number space are circles defined by

$$\left(l + \frac{\beta_0}{2\omega}\right)^2 + m^2 = \left(\frac{\beta_0^2}{4\omega^2} - \frac{1}{R^2}\right), \tag{6-27}$$

and are illustrated in Figure 6-5. Such circles exist only if their radius is a real number—that is, if $\beta_0^2 > 4\omega^2$. This implies the existence of a maximum frequency

$$|\omega|_{\max} = \frac{\beta_0 R}{2}, \tag{6-28}$$

beyond which planetary waves do not exist. The group velocity at which the energy of a wave packet propagates, defined by $(\partial\omega/\partial l,\ \partial\omega/\partial m)$, is the gradient of the function ω in the (l, m) wave-number plane (see Appendix A). It is thus perpendicular to the circles of constant ω. A little algebra reveals that the group-velocity vector is directed inward, toward the center of the circle. Therefore, long waves (small l and m, point near the origin) have westward group velocities, whereas shorter waves (larger l and m, point on the opposite side of the circle) have eastward energy-propagation speeds. This dichotomy is also apparent in Figure 6-4.

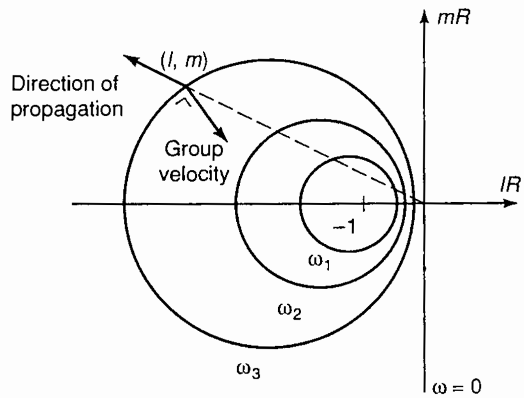


Figure 6-5 Geometric representation of the planetary-wave dispersion relation. Each circle corresponds to a single frequency, with frequency increasing with decreasing radius. The group velocity of the (l, m) wave is a vector perpendicular to the circle at point (l, m) and directed toward its center.

6-5 TOPOGRAPHIC WAVES

Just as small variations in the Coriolis parameter can turn a steady geostrophic flow into slowly moving planetary waves, so can small bottom irregularities. Admittedly, topographic variations can come in a great variety of sizes and shapes, but for the sake of illustrating the wave process in its simplest form, we will content ourselves with the case of a weak and uniform bottom slope. We also return to the use of a constant Coriolis parameter. This latter choice allows us to choose convenient directions for the reference axes, and, in anticipation of an analogy with planetary waves, we align the y-axis with the direction of the topographic gradient. We thus express the depth of the fluid at rest as

$$H = H_0 + \alpha_0 y, \tag{6-29}$$

where H_0 is a mean reference depth and α_0 is the bottom slope, required to be gentle so that

$$\alpha = \frac{\alpha_0 L}{H_0} \ll 1, \tag{6-30}$$

where L is the horizontal length scale of the motion. The *topographic parameter* α plays the role of the planetary number, defined in (6-17).

The bottom slope gives rise to new terms in the continuity equation. Starting with (4-19) and expressing the instantaneous fluid layer depth as (Figure 6-6)

$$h(x, y, t) = H_0 + \alpha_0 y + \eta(x, y, t), \tag{6-31}$$

we obtain

$$\frac{\partial \eta}{\partial t} + \left(u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\right) + (H_0 + \alpha_0 y)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \eta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \alpha_0 v = 0.$$

Again, we strike the nonlinear terms invoking a very small Rossby number (much smaller than the temporal Rossby number) for the sake of linear dynamics. The term

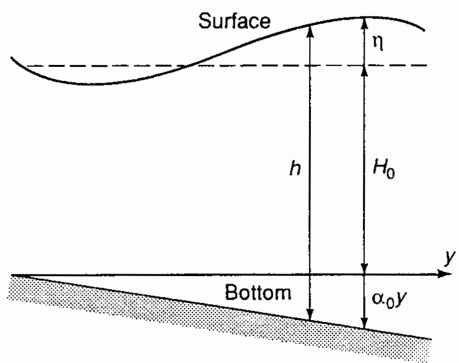


Figure 6-6 A layer of homogeneous fluid over a sloping bottom and the attending notation.

$\alpha_0 y$ can also be dropped as compared to H_0, according to (6-30). With the momentum equations (6-4) and (6-5), our present set of equations is

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}, \tag{6-32a}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}, \tag{6-32b}$$

$$\frac{\partial \eta}{\partial t} + H_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \alpha_0 v = 0. \tag{6-32c}$$

In analogy with the system of equations governing planetary waves, the preceding set contains both small and large terms. The large ones (terms including f, g, and H_0)

comprise the otherwise steady geostrophic dynamics, which correspond to a zero frequency. But, in the presence of the small α_0 term in the last equation, the geostrophic flow cannot remain steady, and the time-derivative terms come into play. We naturally expect them to be small and, compared to the large terms, on the order of α. In other words, the temporal Rossby number, $Ro_T = 1/\Omega T$, is expected to be comparable to α, leading to wave frequencies

$$\omega \sim \frac{1}{T} \sim \alpha\Omega \sim \alpha f \ll f$$

that are very subinertial, just as in the case of planetary waves (where $\omega \sim \beta f_0$).

Capitalizing on the smallness of the time-derivative terms, we take in first approximation the large geostrophic terms: $u \simeq -(g/f)\, \partial\eta/\partial y$, $v \simeq +(g/f)\, \partial\eta/\partial x$. Substitution of these expressions in the small time derivatives yields, to the next degree of approximation:

$$u = -\frac{g}{f}\frac{\partial\eta}{\partial y} - \frac{g}{f^2}\frac{\partial^2\eta}{\partial x\,\partial t} \tag{6-33a}$$

$$v = +\frac{g}{f}\frac{\partial\eta}{\partial x} - \frac{g}{f^2}\frac{\partial^2\eta}{\partial y\,\partial t}. \tag{6-33b}$$

The relative error is only on the order of α^2. Replacement of the velocity components, u and v, by their last expressions [(6-33a) and (6-33b)] in the continuity equation, (6-32c), provides a single equation for the surface displacement η, which to the leading order is

$$\frac{\partial\eta}{\partial t} - R^2\frac{\partial}{\partial t}\nabla^2\eta + \frac{\alpha_0 g}{f}\frac{\partial\eta}{\partial x} = 0. \tag{6-34}$$

(The ageostrophic component v is dropped from the $\alpha_0 v$ term for being on the order of α^2, whereas all other terms are on the order of α.) Note the analogy with equation (6-21) that governs the planetary waves: It is identical, except for the substitution of $\alpha_0 g/f$ for $-\beta_0 R^2$. Here, the deformation radius is defined as

$$R = \frac{\sqrt{gH_0}}{f}, \tag{6-35}$$

that is, the closest constant to the original definition, (6-10). A wave solution of the type $\cos(lx + my - \omega t)$ immediately provides the dispersion relation:

$$\omega = \frac{\alpha_0 g}{f}\,\frac{l}{1 + R^2\,(l^2 + m^2)}, \tag{6-36}$$

the topographic analogue of (6-22). Again, we note that if the additional ingredient, here the bottom slope α_0, had not been present, the frequency would have been nil, and the flow would have been steady and geostrophic. Because they owe their existence to the bottom slope, these waves are called *topographic waves*.

The discussion of their direction of propagation, phase speed, and maximum possible frequency follows that of planetary waves. The phase speed in the x-direction—that is, along the isobaths—is given by

$$c_x = \frac{\omega}{l} = \frac{\alpha_0 g}{f} \frac{1}{1 + R^2\ (l^2 + m^2)} \tag{6-37}$$

and has the sign of $\alpha_0 f$. Thus, topographic waves propagate in the Northern Hemisphere with the shallower side on their right. Because planetary waves propagate westward, or with the north to their right, the analogy between the two kinds of waves is shallow-north and deep-south. (In the Southern Hemisphere, topographic waves propagate with the shallower side on their left, and the analogy is shallow-south, deep-north.)

The phase speed of topographic waves varies with the wave number; they are thus dispersive. The maximum possible wave speed along the isobaths is

$$c = \frac{\alpha_0 g}{f}, \tag{6-38}$$

which is the speed of the very long waves ($l^2 + m^2 \rightarrow 0$). With (6-36) in the form

$$\left(Rl - \frac{\alpha_0 g}{2f\omega R}\right)^2 + (Rm)^2 = \left(\frac{\alpha_0 g}{2f\omega R}\right)^2 - 1,$$

we note that there exists a maximum frequency:

$$|\omega|_{\max} = \frac{|\alpha_0 g|}{|2fR|}. \tag{6-39}$$

The implication is that a forcing at a frequency higher than the preceding threshold cannot generate topographic waves. The forcing generates either a disturbance that is unable to propagate or higher-frequency waves, such as inertia-gravity waves. However, such a situation is rare because, unless the bottom slope is very weak, the maximum frequency given by (6-39) approaches or exceeds the inertial frequency f, and the theory fails before (6-39) can be applied.

As an example, let us take the West Florida Shelf, which is in the eastern Gulf of Mexico. There the ocean depth increases gradually offshore to 200 m over 200 km ($\alpha_0 = 10^{-3}$) and the latitude (27°N) yields $f = 6.6 \times 10^{-5}\ \mathrm{s}^{-1}$. Using an average depth $H_0 = 100$ m, we obtain $R = 475$ km and $\omega_{\max} = 1.6 \times 10^{-4}\ \mathrm{s}^{-1}$. This maximum frequency, corresponding to a mimimum period of 11 min, is larger than f, violates the condition of subinertial motions and is thus meaningless. The wave theory, however, applies to waves whose frequencies are much less than the maximum value; a wavelength of 150 km along the isobaths ($l = 4.2 \times 10^{-5}\ \mathrm{m}^{-1}$, $m = 0$) yields $\omega = 1.6 \times 10^{-5}\ \mathrm{s}^{-1}$ (period of 4.6 days) and a wave speed of $c_x = 0.38$ m/s.

Where the topographic slope is confined between a coastal wall and a flat-bottom abyss, such as for a continental shelf, topographic waves can be trapped, not unlike the Kelvin wave. Mathematically, the solution is not periodic in the offshore, cross-isobath direction but assumes one of several possible profiles (eigenmodes). Each mode has a corresponding frequency (eigenvalue). Such waves are called *continental shelf waves*. The interested reader can find an exposition of these waves in LeBlond and Mysak (1978).

6-6 ANALOGY BETWEEN PLANETARY AND TOPOGRAPHIC WAVES

We have already discussed some of the mathematical similarities between the two kinds of low-frequency waves. The object of this section is to go to the root of the analogy and to compare the physical processes at work in both kinds of waves.

Let us turn to the quantity called potential vorticity and defined in (4-27). On the beta plane and over a sloping bottom (oriented meridionally for convenience), the expression of the potential vorticity becomes

$$q = \frac{f_0 + \beta_0 y + \partial v/\partial x - \partial u/\partial y}{H_0 + \alpha_0 y + \eta}.$$

Our assumptions of a small beta effect and a small Rossby number imply that the numerator is dominated by f_0, all other terms being comparatively very small. Likewise, H_0 is the leading term in the denominator because the bottom slope and the surface displacements are both weak. A Taylor expansion of the fraction yields

$$q = \frac{1}{H_0}\left(f_0 + \beta_0 y - \frac{\alpha_0 f_0}{H_0} y + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{f_0}{H_0}\eta\right). \tag{6-40}$$

We can immediately see that the planetary and topographic terms (β_0 and α_0 terms, respectively) play identical roles. The analogy between the coefficients β_0 and $-\alpha_0 f_0/H_0$ is identical to the one noted earlier between $-\beta_0 R^2$ of (6-22) and $\alpha_0 g/f_0$ of (6-36), since $R = (gH_0)^{1/2}/f_0$. The physical significance is the following: Just as the planetary effect imposes a potential-vorticity gradient, with higher values toward the north, the topographic effect, too, imposes a potential-vorticity gradient, with higher values toward the shallower side.

The presence of such an ambient gradient of potential vorticity is what provides the *bouncing* effect necessary to the existence of the waves. Indeed, consider Figure 6-7, where the first panel represents a north-hemispheric fluid (seen from the top) at rest in a potential-vorticity gradient; think of the fluid as consisting of bands tagged by various potential-vorticity values. The next two panels show the same fluid bands after a wavy disturbance has been applied, in the presence of either the planetary or the topographic effect.

Under the planetary effect (middle panel), fluid parcels caught in crests have been displaced northward and have seen their ambient vorticity, $f_0 + \beta_0 y$, increase; to compensate and conserve their initial potential vorticity, they must develop some negative relative vorticity, that is, a clockwise spin. This is indicated by curved arrows. Similarly, fluid parcels in troughs have been displaced southward, and the decrease of their

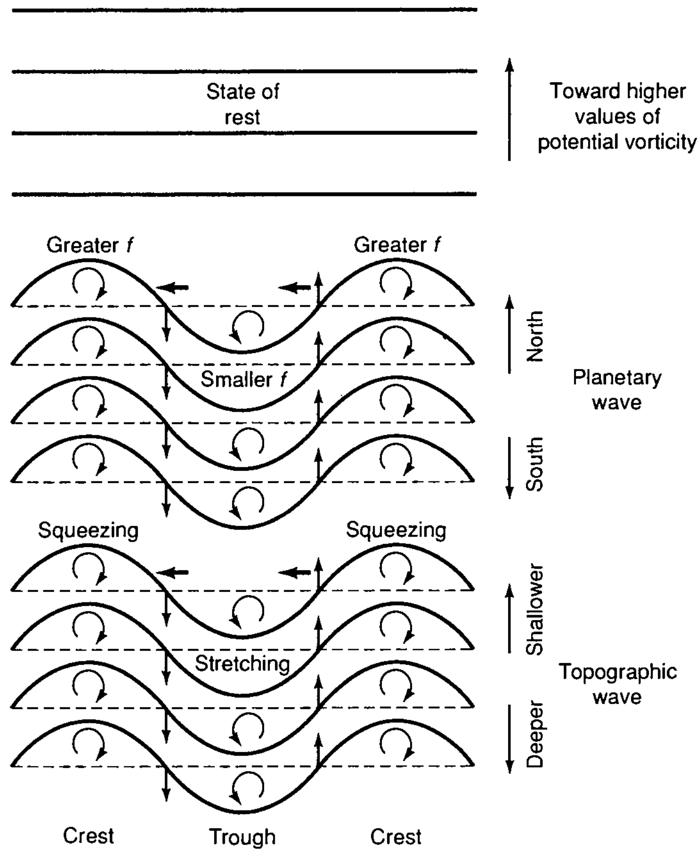


Figure 6-7 Comparison of the physical mechanisms that propel planetary and topographic waves. Displaced fluid parcels react to their new environment by developing either clockwise or counterclockwise vorticity. Intermediate parcels are entrained, and the wave progresses forward.

ambient vorticity is met with an increase of relative vorticity, that is, a counterclockwise spin. Focus now on those intermediate parcels (marked by dots on the figure) that have not been displaced initially; they are sandwiched between two counterrotating vortex patches, and, like an unfortunate finger caught between two gears or the newspaper zipping through the rolling press, they are entrained by the swirling motions and begin to move in the meridional direction. From left to right on the figure, the displacements are southward from crest to trough and northward from trough to crest. Southward displacements set up new troughs whereas northward displacements generate new crests. The net effect is a westward drift of the existing pattern. This explains why planetary waves propagate westward.

In the third panel of Figure 6-7, the preceding exercise is repeated in the case of an ambient potential-vorticity gradient due to a topographic slope. In a crest, a fluid parcel is moved into a shallower environment; the vertical squeezing causes a widening of the parcel's horizontal cross-section (see Section 4-4), which in turn is accompanied by a decrease of relative vorticity. Similarly, parcels in troughs undergo vertical stretching, a lateral narrowing and an increase in relative vorticity. From there on, the story is identical to that of planetary waves. The net effect is a propagation of the trough-crest pattern with the shallow side on the right.

The analogy between the planetary and topographic effects has been found to be extremely useful in the design of laboratory experiments. A sloping bottom in rotating tanks can substitute for the beta effect, which would otherwise be impossible to model experimentally. Caution must be exercised, however, for the substitution is acceptable as long as the analogy holds. Three conditions must be met: absence of stratification, gentle slope, and slow motion. If stratification is present, the sloping bottom will tend to affect preferentially the fluid motions near the bottom, whereas the true beta effect operates evenly at all levels. And, if the slope is not gentle and the motions are not weak, the expression of potential vorticity cannot be linearlized as in (6-40), and the analogy is invalidated.

PROBLEMS

6-1. Prove that Kelvin waves propagate with the coast on their left in the Southern Hemisphere.

6-2. The Yellow Sea between China and Korea (mean latitude: 37°N) has an average depth of 50 m and a coastal perimeter of 2600 km. How long does it take for a Kelvin wave to go around the shores of the Yellow Sea?

6-3. Prove that at extremely large wavelengths, inertia-gravity waves degenerate into a flow field where particles describe circular inertial oscillations.

6-4. An oceanic channel is modeled by a flat-bottom strip of ocean between two vertical walls. Assume that the fluid is homogeneous and inviscid, and that the Coriolis parameter is constant. Describe all waves that can propagate along such channel.

6-5. Consider planetary waves forced by the seasonal variations of the annual cycle. For $f_0 = 8 \times 10^{-5}\ \mathrm{s}^{-1}$, $\beta_0 = 2 \times 10^{-11}\ \mathrm{m}^{-1}\cdot\mathrm{s}^{-1}$, $R = 1000$ km, what is the range of admissible zonal wavelengths?

6-6. Because the Coriolis parameter vanishes along the equator, it is usual in the study of tropical processes to write

$$f = \beta_0 y,$$

where y is the distance measured from the equator (positive northward). The linear wave equations then take the form

$$\frac{\partial u}{\partial t} - \beta_0 y v = -g \frac{\partial \eta}{\partial x},$$

$$\frac{\partial v}{\partial t} + \beta_0 y u = -g \frac{\partial \eta}{\partial y},$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0,$$

where u and v are the zonal and meridional velocity components, η is the surface displacement, g is gravity, and H is the ocean depth. Explore the possibility of a wave traveling zonally with no meridional velocity. At which speed does this wave travel and in which direction? Is it trapped along the equator? If so, what is the trapping distance? Does this wave bear any resemblance to a midlatitude wave (f_0 not zero)?

6-7. Seek wave solutions to the nonhydrostatic system of equations with nonstrictly vertical rotation vector:

$$\frac{\partial u}{\partial t} - fv + f_{*}w = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} - f_{*}u = -\frac{1}{\rho_0}\frac{\partial p}{\partial z}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

The fluid is homogeneous ($\rho = 0$), inviscid ($\nu = 0$) and infinitely deep. Consider in particular the equivalent of the Kelvin wave ($u = 0$ at $x = 0$) and Poincaré waves.

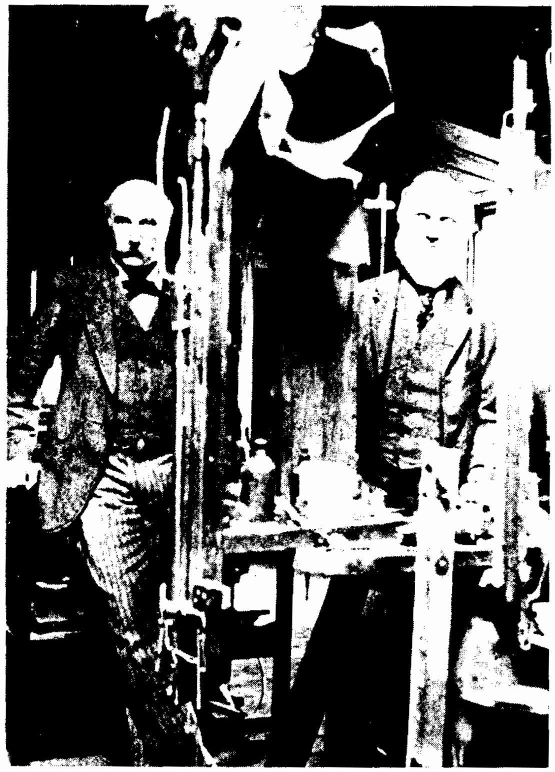
SUGGESTED LABORATORY DEMONSTRATION

Equipment

Rotating tank, water, oil, and an agitator.

Experiment

Fill the rotating tank with water to a depth intermediate between the tank's radius and diameter; then cover the water with a layer of oil. (If the oil is not easily distinguishable from the water, dye the water first.) Select an oil-layer depth such that the internal radius of deformation $[(1 - \rho_{oil}/\rho_{water})gH_{oil}]^{1/2}/2\Omega$ is half the tank's radius or less. Spin the tank gradually to rotation rate Ω and wait for equilibrium to be reached. Then, agitate the oil layer (with a small stick or the like) gently along the tank's periphery. Watch the waves traveling. Note how a slow agitation (frequency less than 2Ω) causes waves that remain along the tank's side (Kelvin waves), whereas quicker agitation (frequency greater than 2Ω) leads to wave radiation across the entire tank (Poincaré waves).



William Thomson, Lord Kelvin

1824 – 1907

(Standing at right, in laboratory of Lord Rayleigh, left)

Named professor of natural philosophy at the University of Glasgow, Scotland, at age 22, William Thomson became quickly regarded as the leading inventor and scientist of his time. In 1892, he was named Baron Kelvin of Largs for his technological and theoretical contributions leading to the successful laying of the transatlantic cable. A friend of Joule's, he helped establish a firm theory of thermodynamics and first defined the absolute scale of temperature. He also made major contributions to the study of heat engines. With Hermann von Helmholtz, he estimated the ages of the earth and sun and ventured in fluid mechanics (see Figures 11-2 and 11-3). His theory of the so-called Kelvin wave was published in 1879 (under the name William Thomson). His more than 300 original papers left hardly any aspect of science untouched. He is quoted as saying that he could understand nothing of which he could not make a model. (*Photo by A. G. Webster.*)