7

Barotropic Instability

Summary: The waves explored in the previous chapter evolve in a fluid otherwise at rest, propagating without either growth or decay. Here, we investigate waves riding on an existing current and find that, under certain conditions, they may grow at the expense of the energy contained in the mean current.

7-1 INTRODUCTION

The planetary and topographic waves described in Chapter 6 (Sections 6-4 through 6-6) owe their existence to the presence of an ambient potential-vorticity gradient. In the case of planetary waves, the cause is the sphericity of the planet, whereas for topographic waves the gradient results from the bottom slope. We may naturally wonder whether a sheared current, with a weak gradient in relative vorticity, would not be able to sustain similar low-frequency waves.

The situation is quite different, however, for several reasons. First, the current would not only create the required ambient potential-vorticity gradient but would also transport the wave pattern; because of the current shear, this translation would be differential, and the wave pattern would be rapidly distorted. Moreover, there is likely to be a place within the current where the phase speed of the wave matches that of the mean current; such a location, termed the *critical level*, typically permits a vigorous transfer of energy between the basic current and the wave. As a consequence, the wave may draw energy from the current and grow in time. If this happens, insignificant little wiggles may turn into very large perturbations, and the basic flow is highly contorted to the point of becoming unrecognizable. The current is said to be unstable. To distinguish this situation from other instabilities occurring in baroclinic fluids (i.e., those possessing a stratification; see Chapter 16), the preceding process is generally known as *barotropic instability*.

The stability theory of homogeneous shear flows is a well-developed chapter in fluid mechanics (see, for example, Kundu, 1990). Here, we generalize the problem by introducing the Coriolis force, but limit our investigation to establishing general properties and solving a particular case.

7-2 WAVES ON A SHEAR FLOW

To investigate the behavior of waves on an existing current in a relatively clear and tractable formalism, it is customary to make the following assumptions: The fluid is homogeneous and inviscid, and the bottom and the surface are flat and horizontal. The Coriolis parameter is, however, allowed to vary (i.e., the beta effect is retained). The governing equations are (Section 3-5)

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\,\frac{\partial p}{\partial x} \tag{7-1}$$

$$\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\,\frac{\partial p}{\partial y} \tag{7-2}$$

$$0 = -\frac{\partial p}{\partial z} \tag{7-3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{7-4}$$

where the Coriolis parameter $f = f_0 + \beta_0 y$ varies with the northward coordinate y (Section 6-4). As demonstrated in Section 4-3, a horizontal flow that is initially uniform in the vertical will, in the absence of vertical friction, remain so at all times. We consider such a case and, consequently, drop the terms $\partial u/\partial z$ and $\partial v/\partial z$ in equations (7-1) and (7-2), respectively. According to (7-4), $\partial w/\partial z$ must be z-independent, and w is linear in z. But, because the vertical velocity vanishes at both top and bottom, it must be zero everywhere. The continuity equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{7-5}$$

For the basic state, we choose a zonal current with arbitrary meridional profile:

$u = \bar{u}(y)$, $v = 0$. This is an exact solution to the nonlinear equations as long as the pressure profile, $p = \bar{p}(y)$, satisfies

$$(f_0 + \beta_0 y)\, \bar{u}(y) = -\frac{1}{\rho_0}\frac{d\bar{p}}{dy}. \tag{7-6}$$

Next, we add a small perturbation, meant to represent an arbitrary wave of weak amplitude. We write

$$u = \bar{u}(y) + u'(x, y, t)$$
$$v = v'(x, y, t)$$
$$p = \bar{p}(y) + p'(x, y, t),$$

where the perturbations u', v', and p' are taken to be much smaller that those of the basic flow (i.e., u' and v' much less than $\bar{u}$, and p' much less than $\bar{p}$). Substitution in equations (7-1), (7-2), and (7-5) and subsequent linearization that takes advantage of the smallness of the perturbation yield

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + v'\frac{d\bar{u}}{dy} - (f_0 + \beta_0 y)\, v' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \tag{7-7}$$

$$\frac{\partial v'}{\partial t} + \bar{u}\frac{\partial v'}{\partial x} + (f_0 + \beta_0 y)\, u' = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \tag{7-8}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0. \tag{7-9}$$

The last equation admits the streamfunction ψ, defined as

$$u' = -\frac{\partial \psi}{\partial y}, \qquad v' = +\frac{\partial \psi}{\partial x}. \tag{7-10}$$

The choice of signs corresponds to a flow along streamlines, with the higher streamfunction values on the right.

A cross-differentiation of the momentum equations (7-7) and (7-8) and the elimination of the velocity components leads to a single equation for the streamfunction:

$$\left(\frac{\partial}{\partial t} + \bar{u}\frac{\partial}{\partial x}\right)\nabla^2\psi + \left(\beta_0 - \frac{d^2\bar{u}}{dy^2}\right)\frac{\partial \psi}{\partial x} = 0. \tag{7-11}$$

This equation has coefficients that depend on $\bar{u}$ and, therefore, on the meridional coordinate y only. A sinusoidal wave in the zonal direction is a solution: $\psi(x, y, t) = \phi(y)\exp[i(lx - \omega t)]$. Substitution provides the following second-order ordinary differential equation for the amplitude $\phi(y)$:

$$\frac{d^2\phi}{dy^2} - l^2\phi + \frac{\beta_0 - d^2\bar{u}/dy^2}{\bar{u}(y) - c}\,\phi = 0, \tag{7-12}$$

where $c = \omega/l$ is the zonal speed of propagation. An equation of this type is called a *Rayleigh equation*.

Little else can be said without the introduction of boundary conditions. For simplicity, let us assume that the fluid is contained between two vertical walls at $y = 0$ and L. We are thus considering waves on a zonal flow in a zonal channel. Obviously, there is no such zonal channel in the atmosphere and oceans, but wavy zonal flows of limited meriodional extent abound. The atmospheric jet stream in the upper troposphere, the Gulf Stream after its seaward turn off Cape Hatteras (36°N), and the Antarctic Circumpolar Current are all good examples. Also, the atmosphere on Jupiter, with the exception of the Great Red Spot and other vortices, consists almost entirely of zonal bands of alternating winds.

If the boundaries prevent fluid from entering and leaving the channel, v' is zero there, and (7-10) implies that the streamfunction must take constant values along each wall. (The walls are streamlines.) This is possible only if the wave amplitude obeys

$$\phi(y = 0) = \phi(y = L) = 0 . \tag{7-13}$$

The second-order, homogeneous problem of (7-12) and (7-13) can be viewed as an eigenvalue problem: The solution is trivial ($\phi = 0$), unless the phase velocity assumes a specific value (eigenvalue), in which case a nonzero function ϕ can be determined within an arbitrary multiplicative constant.

In general, the eigenvalues c may be complex. If c admits the function ϕ, then the complex conjugate c^* admits the complex conjugate function ϕ^* and is thus another eigenvalue. This can be readily verified by taking the complex conjugate of equation (7-12). Hence, complex eigenvalues come in pairs.

Decomposing the eigenvalue into its real and imaginary components, $c = c_r + ic_i$, we note that the streamfunction ψ has an exponential factor of the form $\exp(lc_i t)$, which grows or decays according to the sign of c_i. Because the eigenvalues come in pairs, to any decaying mode will correspond a growing mode. Therefore, the presence of a nonzero imaginary part in the phase velocity c automatically guarantees the existence of a growing disturbance and thus the instability of the basic flow. The product lc_i is then called the *growth rate*. Conversely, the basic flow is stable if and only if the phase speed is purely real.

Because it is impossible in general to determine the values c for an arbitrary velocity profile $\bar{u}(y)$ (the analysis is difficult even for specific but nontrivial profiles), we shall not attempt to solve the problem in (7-12) and (7-13) but will instead establish some of its integral properties and, in so doing, reach a weaker stability criterion.

If we multiply equation (7-12) by ϕ^* and integrate across the domain, we get

$$-\int_0^L \left(\left| \frac{d\phi}{dy} \right|^2 + l^2 |\phi|^2 \right) dy + \int_0^L \frac{\beta_0 - d^2\bar{u}/dy^2}{\bar{u} - c} \, |\phi|^2 \, dy = 0 , \tag{7-14}$$

after an integration by parts. The imaginary part of this expression is

$$c_i \int_0^L \left(\beta_0 - \frac{d^2\bar{u}}{dy^2} \right) \frac{|\phi|^2}{|\bar{u} - c|^2} \, dy = 0 . \tag{7-15}$$

Two cases are possible: Either c_i vanishes or the integral does. If c_i is zero, the basic flow admits no growing disturbance and is stable. If c_i is not zero, then the integral must vanish, which requires that the quantity

$$\beta_0 - \frac{d^2\bar{u}}{dy^2} = \frac{d}{dy}\left(f_0 + \beta_0 y - \frac{d\bar{u}}{dy}\right) \tag{7-16}$$

must change sign at least once within the confines of the domain. Summing up, we conclude that a necessary condition for instability is that expression (7-16) vanish somewhere inside the domain. Conversely, a sufficient condition for stability is that expression (7-16) not vanish anywhere within the domain (on the boundaries maybe, but not inside the domain). Physically, the total vorticity of the basic flow, $f_0 + \beta_0 y - d\bar{u}/dy$, must reach an extremum within the domain to cause instabilities. This result was first derived by Kuo (1949).

This first criterion can be strengthened by considering next the real part of (7-14):

$$\int_0^L (\bar{u} - c_r)\left(\beta_0 - \frac{d^2\bar{u}}{dy^2}\right)\frac{|\phi|^2}{|\bar{u} - c|^2}\, dy = \int_0^L \left(\left|\frac{d\phi}{dy}\right|^2 + l^2|\phi|^2\right) dy. \tag{7-17}$$

In the event of instability, the integral in (7-15) vanishes. Multiplying it by $(c_r - \bar{u}_0)$, where $\bar{u}_0$ is any real constant, adding the result to (7-17), and noting that the right-hand side of (7-17) is always positive for nonzero perturbations, we obtain

$$\int_0^L (\bar{u} - \bar{u}_0)\left(\beta_0 - \frac{d^2\bar{u}}{dy^2}\right)\frac{|\phi|^2}{|\bar{u} - c|^2}\, dy > 0.$$

This inequality demands that the expression

$$(\bar{u} - \bar{u}_0)\left(\beta_0 - \frac{d^2\bar{u}}{dy^2}\right) \tag{7-18}$$

be positive in at least some finite portion of the domain. Because this must hold true for any constant $\bar{u}_0$, it must be true in particular if $\bar{u}_0$ is the value of $\bar{u}(y)$ where $\beta_0 - d^2\bar{u}/dy^2$ vanishes. Hence, a stronger criterion is: Necessary conditions for instability are that $\beta_0 - d^2\bar{u}/dy^2$ vanish at least once within the domain *and* that $(\bar{u} - \bar{u}_0)(\beta_0 - d^2\bar{u}/dy^2)$, where $\bar{u}_0$ is the value of $\bar{u}(y)$ at which the first expression vanishes, be positive in at least some finite portion of the domain. Although this stiffer criterion still offers no sufficient condition for instability, it is generally quite useful.

7-3 BOUNDS ON WAVE SPEEDS AND GROWTH RATES

The preceding analysis taught us that instabilities may occur, if certain conditions are met. A question then naturally arises: If the flow is unstable, how fast will perturbations grow? In the general case of an arbitrary shear flow $\bar{u}(y)$, a precise determination of the growth rate of unstable perturbations is obviously not possible. However, an upper bound can be derived relatively easily; in the process, we can even set lower and upper bounds on the phase speed of the perturbations. For simplicity, we will restrict our attention to the f-plane ($\beta_0 = 0$), in which case the derivation is due to Howard (1961). Afterward, we will cite, without demonstration, the result for the β-plane.

The analysis begins by a change of variable. Instead of the streamfunction, let us take the meridional displacement, d, of a fluid parcel, defined from the meridional velocity, v,

$$v = \frac{\partial d}{\partial t} + u \frac{\partial d}{\partial x} + v \frac{\partial d}{\partial y}. \tag{7-19}$$

In accordance with our previous assumption of weak perturbations on an otherwise zonal shear flow $\bar{u}(y)$ and with our definition of the streamfunction, we have

$$\frac{\partial \psi}{\partial x} = \frac{\partial d}{\partial t} + \bar{u} \frac{\partial d}{\partial x},$$

or, after the elimination of the x and t variables by the introduction of the Fourier mode $\{\psi, d\} = \{\phi(y), a(y)\} \exp[il(x - ct)]$,

$$\phi = (\bar{u} - c)a. \tag{7-20}$$

Substitution in equation (7-12) yields the following equation for the amplitude of the meridional displacement:

$$\frac{d}{dy}\left[(\bar{u} - c)^2 \frac{da}{dy}\right] - l^2(\bar{u} - c)^2 a = 0, \tag{7-21}$$

with β_0 set to zero. Because of (7-20), the boundary conditions on a are identical to those on ϕ, namely, $a(0) = a(L) = 0$.

We consider the case of an unstable wave. In this case, c has a nonzero imaginary part, and a is complex and nonzero. Multiplying by the complex conjugate a^* and integrating across the domain, we obtain an expression whose real and imaginary parts are as follows:

$$\text{Real part:} \qquad \int_0^L [(\bar{u} - c_r)^2 - c_i^2] P \, dy = 0 \tag{7-22}$$

$$\text{Imaginary part:} \qquad \int_0^L (\bar{u} - c_r) P \, dy = 0, \tag{7-23}$$

where $P = |da/dy|^2 + l^2|a|^2$ is a nonzero positive quantity. From (7-23), it immediately follows that $(\bar{u} - c_r)$ must vanish somewhere in the domain, implying that the phase speed c_r lies between the minimum and maximum values of $\bar{u}(y)$:

$$U_{\min} < c_r < U_{\max}. \tag{7-24}$$

Physically, the wavy perturbation, if unstable, must travel with a speed that matches that of the entraining flow, in at least one location. In other words, there will always be a place in the domain where the wave does not drift with respect to the ambient flow and grows in place. It is precisely this local coupling between wave and flow that allows the wave to extract energy from the flow and to grow at its expense. The location where the phase speed is equal to the flow velocity is called a *critical level.*

Armed with bounds for the real part of c, we now seek bounds on its imaginary part. To do so, we introduce the obvious inequality

$$\int_0^L (\bar{u} - U_{\min})\ (U_{\max} - \bar{u})P\ dy \geq 0 \tag{7-25}$$

and then add it to equation (7-22). From the result, we subtract (7-23) premultiplied by $(U_{\min} + U_{\max} - 2c_r)$ and rearrange terms to obtain

$$\left[\left(c_r - \frac{U_{\min} + U_{\max}}{2}\right)^2 + c_i^2 - \left(\frac{U_{\max} - U_{\min}}{2}\right)^2\right] \int_0^L P\ dy \leq 0 .$$

Because the integral can only be positive, the preceding bracketed quantity must be negative:

$$\left(c_r - \frac{U_{\min} + U_{\max}}{2}\right)^2 + c_i^2 \leq \left(\frac{U_{\max} - U_{\min}}{2}\right)^2 . \tag{7-26}$$

This inequality implies that, in the complex plane, the number $c_r + ic_i$ must lie within the circle centered at $((U_{\min} + U_{\max})/2, 0)$ and of radius $(U_{\max} - U_{\min})/2$. Since we are interested in modes that grow in time, c_i is positive, and only the upper half of that

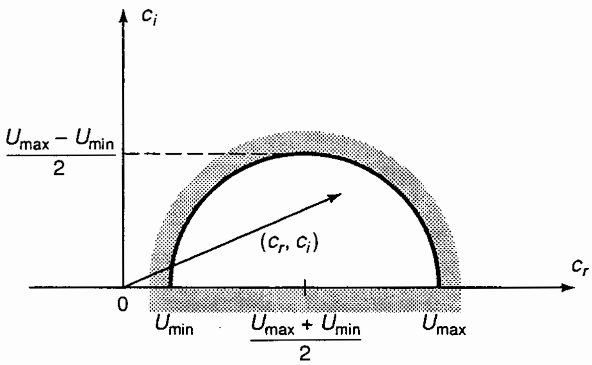

Figure 7-1 The semicircle theorem. Growing perturbations of wave number l must have phase speeds c_r and growth rates lc_i such that the end point of the vector (c_r, c_i) must lie within the half-circle constructed from the minimum and maximum velocities of the ambient shear flow, as depicted in the figure.

circle is relevant (Figure 7-1). This result is called Howard's semicircle theorem. It is readily evident from inequality (7-26) or Figure 7-1 that c_i is bounded above by

$$c_i \leq \frac{U_{\max} - U_{\min}}{2} . \tag{7-27}$$

The perturbation's growth rate lc_i is thus likewise bounded above.

On the beta plane, the treatment of integrals and inequalities is a little more elaborate but still feasible. It can then be shown (Pedlosky, 1987, Section 7-5) that the preceding inequalities on c_r and c_i must be modified to

$$U_{\min} - \frac{\beta_0 L^2}{2(\pi^2 + l^2 L^2)} < c_r < U_{\max} \tag{7-28}$$

$$\left(c_r - \frac{U_{\min} + U_{\max}}{2}\right)^2 + c_i^{\,2} \leq \left(\frac{U_{\max} - U_{\min}}{2}\right)^2 + \frac{\beta_0 L^2 (U_{\max} - U_{\min})}{2(\pi^2 + l^2 L^2)}, \tag{7-29}$$

where L is the domain's meridional width and l is the zonal wave number. The westward velocity shift on the left side of (7-28) is related to the existence of planetary waves [see the zonal phase speed, (6-25)]. The last inequality readily leads to an upper bound for the growth rate lc_i.

7-4 A SIMPLE EXAMPLE

The preceding considerations on the existence of instabilities and their properties are rather abstract. So, let us work out an example to illustrate the concepts. For simplicity,

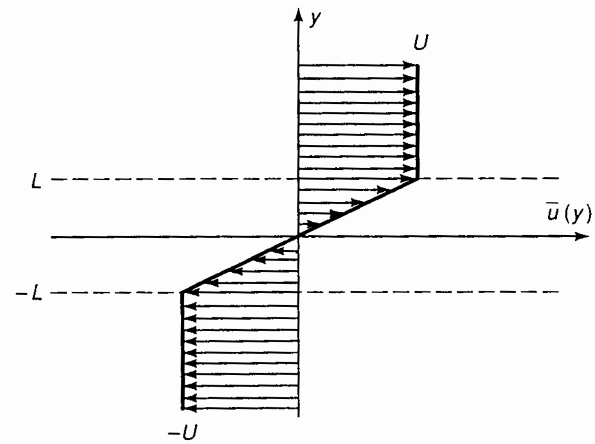

Figure 7-2 An academic shear-flow profile that lends itself to analytic treatment. This profile meets both necessary conditions for instability and is found to be unstable to long waves.

we again restrict ourselves to the f-plane ($\beta_0 = 0$), and we take a shear flow that is piecewise linear (Figure 7-2):

$$y < -L: \qquad \bar{u} = -U, \quad \frac{d\bar{u}}{dy} = 0, \quad \frac{d^2\bar{u}}{dy^2} = 0$$

$$-L < y < +L: \qquad \bar{u} = \frac{U}{L}\, y, \quad \frac{d\bar{u}}{dy} = \frac{U}{L}, \quad \frac{d^2\bar{u}}{dy^2} = 0$$

$$+L < y: \qquad \bar{u} = +U, \quad \frac{d\bar{u}}{dy} = 0, \quad \frac{d^2\bar{u}}{dy^2} = 0,$$

where U is a positive constant and the domain width is now infinity. Although the second derivative vanishes within each of the three segments of the domain, it is nonzero at their junctions. As y increases, the first derivative $d\overline{u}/dy$ changes from zero to a positive value back to zero, so it can be said that the second derivative is positive at the first junction ($y = -L$) and negative at the second ($y = L$). It thus changes sign in the domain, and this satisfies the first condition for the existence of instabilities, namely, that $d^2\overline{u}/dy^2$ must vanish within the domain. The second condition, which requires that expression (7-18), now reduced to

$$-\overline{u}\,\frac{d^2\overline{u}}{dy^2},$$

be positive in some portion of the domain, is also satisfied because $d^2\overline{u}/dy^2$ has the sign opposite to $\overline{u}$ at each junction of the profile. So, although instabilities are not guaranteed to exist, we may not rule them out.

We now proceed with the solution. In each of the three domain segments, governing equation (7-12) reduces to

$$\frac{d^2\phi}{dy^2} - l^2\,\phi = 0,$$

and admits solutions of the type $\exp(+\,ly)$ and $\exp(-\,ly)$. This yields two constants of integration per domain segment, for a total of six. Six conditions are then applied. First, ϕ is required to vanish at large distances:

$$\phi(-\infty) = \phi(+\infty) = 0.$$

Next, continuity of the meridional displacements at $y = \pm L$ requires, by virtue of (7-20) and by virtue of the continuity of the $\overline{u}(y)$ profile, that ϕ, too, be continuous there:

$$\phi(-L-\varepsilon) = \phi(-L+\varepsilon), \qquad \phi(+L-\varepsilon) = \phi(+L+\varepsilon),$$

for arbitrarily small values of ε. Finally, the integration of governing equation (7-12) arbitrarily across the lines joining the domain segments

$$\int_{\pm L-\varepsilon}^{\pm L+\varepsilon}\left[(\overline{u}-c)\,\frac{d^2\phi}{dy^2} - l^2(\overline{u}-c)\,\phi - \frac{d^2\overline{u}}{dy^2}\,\phi\right]dy = 0,$$

followed by an integration by parts, implies that

$$(\overline{u}-c)\,\frac{d\phi}{dy} - \frac{d\overline{u}}{dy}\phi$$

must be continuous at both $y = -L$ and $y = +L$. Applying these six conditions leads to a homogeneous system of equations for the six constants of integration. Nonzero perturbations are found when this system admits a nontrivial solution—that is, when its determinant vanishes. Some tedious algebra yields

$$\frac{c^2}{U^2} = \frac{(1-2lL)^2 - e^{-4lL}}{(2lL)^2}. \tag{7-30}$$

Equation (7-30) is the dispersion relation, providing the wave velocity c in terms of the wave number l and the flow parameters L and U. It yields a unique and real c^2, either positive or negative. If it is positive, c is real and the perturbation behaves as a nonamplifying wave. But, if c^2 is negative, c is imaginary and one of the two solutions yields an exponentially growing mode [a proportional to $\exp(lc_i t)$]. Obviously, the instability threshold is $c^2 = 0$, in which case the dispersion relation (7-30) yields $lL = 0.639$. There is thus a critical wave number $l = 0.639/L$ or critical wavelength $2\pi/l = 9.829L$ separating stable from unstable waves. It can be shown by inspection of the dispersion relation that shorter waves ($lL > 0.639$) travel without growth, whereas longer waves ($lL < 0.639$) grow without bound. In conclusion, the basic shear flow is unstable to long-wave disturbances.

At this point, it is instructive to unravel the physical mechanism responsible for the growth of long-wave disturbances. Figure 7-3 displays the basic flow field, on which

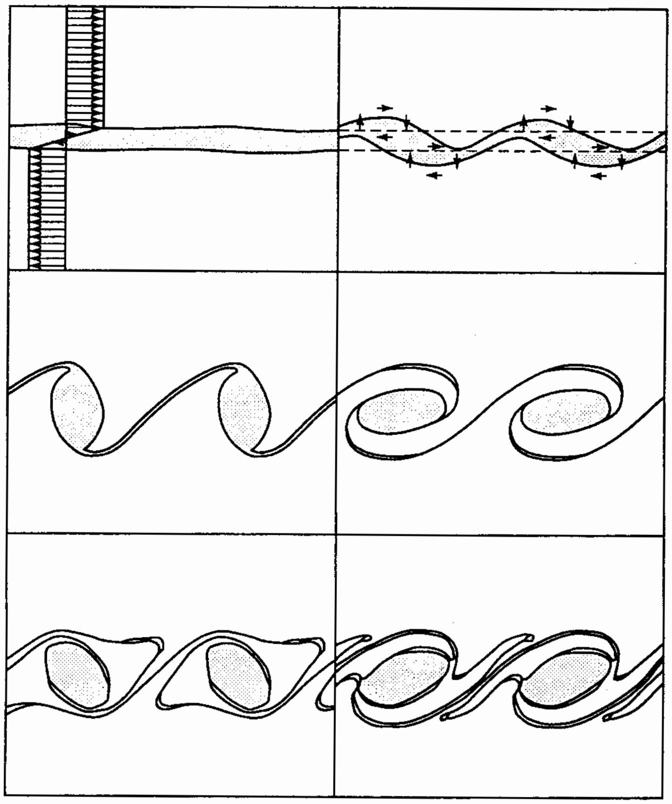

Figure 7-3 Physical interpretation of the development of wavy disturbances on the shear flow of Figure 7-2. The troughs and crests of the wave induce a vortex field, which, in turn, amplifies those troughs and crests. The wave does not travel but amplifies with time. (Adapted from Dritschel, 1989.)

is superimposed a wavy disturbance. The phase shift between the two lines of discontinuity is that propitious to wave amplification. As the middle fluid, endowed with a clockwise vorticity, intrudes in either neighboring strip, where the vorticity is nonexistent, it produces local vorticity anomalies, which can be viewed as vortices. These vortices generate clockwise rotating flows in their vicinity, and, if the wavelength is sufficiently long, the distance between the two lines of discontinuity appears relatively small and the vortices from each side interact with those on the other side. Under a proper phase difference, such as the one depicted on Figure 7-3, the vortices entrain one another further into the regions of no vorticity, thereby amplifying the crests and troughs of the wave. The wave amplifies, and the basic shear flow cannot persist. As the wave grows, nonlinear terms are no longer negligible, and some level of saturation is reached. The ultimate state (Figure 7-3) is that of a series of clockwise vortices embedded in a weakened ambient shear flow (Zabusky et al., 1979; Dritschel, 1989).

PROBLEMS

7-1. What can you say of the stability properties of the following flow fields on the f-plane?

$$\bar{u}(y) = U\left(1 - \frac{y^2}{L^2}\right) \quad (-L \leq y \leq +L),$$

$$\bar{u}(y) = U \sin\frac{\pi y}{L} \quad (0 \leq y \leq L),$$

$$\bar{u}(y) = U \cos\frac{\pi y}{L} \quad (0 \leq y \leq L),$$

$$\bar{u}(y) = U \tanh\left(\frac{y}{L}\right) \quad (-\infty < y < +\infty).$$

7-2. A zonal shear flow with velocity profile

$$\bar{u}(y) = U\left(\frac{y}{L} - 3\,\frac{y^3}{L^3}\right)$$

occupies the channel $-L \leq y \leq +L$ on the beta plane. Show that if $|U|$ is less than $\beta_0 L^2/12$, this flow is stable.

7-3. The atmospheric jet stream is a wandering zonal flow of the upper troposphere, which, by and large, determines our weather. If we ignore the variations in air density, we can model the average jet stream as a purely zonal flow, independent of height and varying meridionally according to

$$\bar{u}(y) = U \exp\left(-\frac{y^2}{2L^2}\right),$$

where the constants U and L, characteristics of the speed and width, respectively, are taken as 40 m/s and 570 km. The jet center ($y = 0$) is at 45°N where $\beta_0 = 1.61 \times 10^{-11}$ $\mathrm{m^{-1} \cdot s^{-1}}$. Is the jet stream unstable to shear waves?

7-4. Verify the semicircle theorem for the particular shear flow discussed in Section 7-4. In other words, prove that $|c_r| < U$ for stable waves and $c_i < U$ for unstable waves. Also, for which wavelength is the growth rate, lc_i, maximum?

7-5. Derive the dispersion relation and establish a precise threshold of stability for the jetlike profile of Figure 7-4.

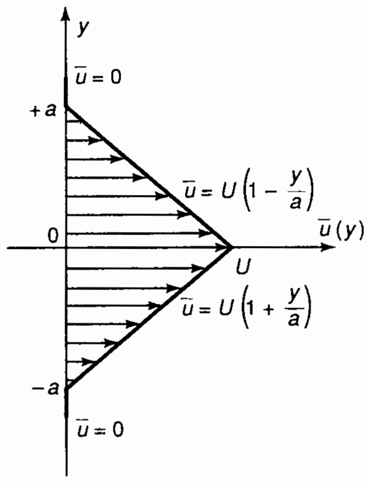

Figure 7-4 A jetlike profile (for Problem 7-5).

Louis Norberg Howard

1929 –

Applied mathematician and fluid dynamicist, Louis Norberg Howard has made numerous contributions to hydrodynamic stability and rotating flow. His famous semicircle theorem was published in 1961 as a short note extending some contemporary work by John Miles. Howard is also well known for his theoretical and experimental studies of natural convection. With Willem Malkus, he devised a simple waterwheel model of convection, which, like real convection, can exhibit resting, steady, periodic and chaotic behaviors. Until 1984, Howard was a regular lecturer at the annual Geophysical Fluid Dynamics Summer Institute at the Woods Hole Oceanographic Institution, where his audiences were much impressed by the breadth of his knowledge and the clarity of his explanations. (*Photo credit: L. N. Howard.*)