CMS 807 Final Exam Spring 1988 Andreas Munchow 23/05/88

Intro, 93 Well done, even if not concise.

Pedlosky's (1987) equation (6.8.9)

$$\frac{d_0}{dt}\left[\zeta_0 + \beta y - \frac{\partial}{\partial z}\frac{\rho_0}{S}\right] = 0$$

derives from the "...precisely ~~exact~~ $O(1)$ vorticity equation ..." assuming a flat earth with the Coriolis parameter varying linearily with latitude (6.3.17)

$$\frac{d_0}{dt}\left(\zeta_0 + \beta y\right) = \frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho w_1)$$ OK

The latter equation represents quasi-geostrophic dynamics in its full representation, the former equation is an approximation of the latter. Both equations, their validity, and physical meaning and interpretation are discussed subsequently.

The independent variables are (x, y, z, t), the dependent variables are

Is this zeta? I doubt

$$\zeta_0 = \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}$$ i.e. v_0, u_0 horizontal geostrophic velocities

$$\rho = \rho_s(z) + \rho'(x,y,z,t) \quad \text{and} \quad \rho' = \rho_0 + \varepsilon\rho_1 + \dots$$

ρ_s is a known background density state and ρ' is a density perturbation due to the motion.

The horizontal velocities to be geostrophic means that to $O(1)$ in an ε-expansion ($\varepsilon = U/2\Omega L$, the Rossby #) (u_0, v_0, p_0) satisfy

$$v_0 = \frac{\partial p_0}{\partial x}, \quad u_0 = -\frac{\partial p_0}{\partial y} \quad ✓$$

and the vertical hydrostatic equation

$$\rho_0 = -\frac{1}{\rho_s}\frac{\partial}{\partial z}(\rho_s p_0) \quad ✓$$

Pedlosky (6.3.6)

Hence $\zeta_0 = \nabla_H p_0$, i.e. all three dependent variables can be expressed in terms of p_0, the first order pressure perturbation. An expression for thermodynamic processes enters the dynamical description when the $O(\varepsilon)$ vertical velocity w_1 in (6.3.17) is expressed in terms of p_0 only ✓. S and β are nondimensional parameters discussed later.

Then, (6.8.9) is a nonlinear (because of $\frac{d_0}{dt} = \frac{\partial}{\partial t} + u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}$) partial differential equation in (x, y, z, t) for the dependent variable p_0 only. All other dependent quantities can then be solved for once p_0 is known.

This question has a <u>one</u> line answer: Pot. Vort. (or angular momentum) is conserved following the hor. motion.

(1)

The equation

$$\frac{d_0}{dt}\left[\xi_0 + \beta y - \frac{\partial}{\partial t}\frac{p_0}{S}\right] = 0$$

conservation of what?

is a conservation statement following a geostrophically ~~stream~~ moving particle, i.e. for a particle moving with (u_0, v_0) the quantity

$$\xi_0 + \beta y - \frac{\partial}{\partial t}\frac{p_0}{S}$$

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is a constant and stays constant for that particle as it moves.

The first term ξ_0 is its relative vorticity, i.e. its ~~local shear~~ $\left(\xi_0 = \vec{k}\cdot\nabla\times\vec{u}_0\} = \frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}\right)$ local vorticity.

The second term βy represents the planetary vorticity, i.e. the effects of the earth's rotation upon the particle, and the last term $-\frac{\partial}{\partial t}\, p_0/S$ represents vortex line stretching or squashing due to variations of isopycnal surfaces.

Hence it is the last term which introduces the effect of a variable density stratification.

Because the equation is ~~too~~ scaled it states (~~for~~ $\beta = O(1)$ and $S = O(1)$) that the contribution of ~~relative~~ a particles vorticity by relative vorticity, by absolute vorticity, and by vortex line stretching due to density variations are of the same importance and its sum stays constant as the particle moves with (u_0, v_0)

(2) The assumptions to derive the precise quasi-geostrophic potential vorticity equation (6.3.17) are:

(i) Small Rossby # (~~large so~~

$$\varepsilon = \frac{U}{2\Omega L} \ll 1$$ meaning?

(ii) β-plane, i.e. planetary vorticity varies linearly with latitude

$$\frac{L}{r_0} = \mathcal{O}(\varepsilon)$$

r_0 the radius of the earth

L a lengthscale of the motion

(iii) Vortex tube stretching due to the free surface is negligible

$$F = \left(\frac{L}{R_0}\right)^2 = \left(\frac{L}{\sqrt{gD}/f_0}\right)^2 = \mathcal{O}(\varepsilon)$$

$\sqrt{gD}/f_0$ is the barotropic Rossby radius of deformation. D is a depth scale, $f_0 = 2\Omega$ the local ~~Coriolis~~ rate of earth's rotation

(iv) We stay away from frictional boundaries, i.e. interior dynamics only:

$$(E_V, E_H) = \mathcal{O}(\varepsilon)$$

E_V, E_H are vertical and horizontal Ekman numbers

(v) To ensure hydrostatic balance, we require $\delta = \frac{D}{L} \ll 1$

Another additional assumption is needed to derive equation (3.8.9) from the precise vorticity equation (6.3.17):

(vi) The scale of motion is of the order of the baroclinic Rossby radius of deformation L_I (mesoscale, synoptic scale), i.e.,

$$S = \left(\frac{L_I}{L}\right)^2 = \left(\frac{\sqrt{g' D^{*}}/f_0}{L}\right)^2 = O(1)$$

g' is a reduced gravity $\frac{\Delta\rho}{\rho} g$ indicating the gravity effects of different density 'layers', say. D, then, is a typical layer depth rather than the total depth.

(vii) Neglect internal heating

Crucial in facilitating the thermodynamics to find a simple model of how the thermodynamics enter the quasi-geostrophic dynamics is condition (vi), which can also be restated that the buoyancy frequency N_s (Brunt Väisälä frequency) is small

$$N_s = \left(-\frac{g}{\rho}\frac{\partial \rho_s}{\partial z_*}\right)^{1/2} \quad \text{in dimensional form}$$

also, $D/H_* \ll 1$ and $h_B/D \leq O(\varepsilon)$

as $S^{1/2} = \cancel{\frac{N_s D^2}{fL}} \quad \frac{N_s D}{f L}$

However, the dynamical important length scale ($S = O(1)$) is L_D, the meso scale.

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(3) The physical interpretation of $\frac{d_0}{dt}\,\xi_0$ is

the geostrophical rate of change of the relative vorticity ξ_0, i.e., to be important the horizontal geostrophic velocities have to change rapidly enough, because potentially this term might be balanced by the β effect, i.e. $\beta \frac{d_0}{dt} y$ that is as the individual particle moving with (u_0, v_0) changes its latitude (its position northward, y) its planetary vorticity increases or decreases. On the oceanic scale, $L \sim 2000$ km we usually neglect the relative vorticity present to be small, but here our spatial length scale is different (baroclinic Rossby radius instead of the barotropic one) and therefore the relative vorticity is an $O(1)$ contribution in the vorticity balance (potentially). The dynamical flow field resulting rapid spatial changes in an eddy field, the particles may enter and leave eddies, changing their rel. vorticity according to their position. The term $-\frac{1}{S}\frac{d_0}{dt}\left(\frac{\partial p_0}{\partial z}\right)$ indicates that relative and planetary vorticity might be balanced by changing ~~the~~ isopycnals ($\rho_0 = $ const) vertically as the fluid particle moves along. It represents vortex tube stretching and squashing because ($\rho_0 = $ const) is a ~~three dimensional~~ surface varying spatially its depth. The particle with a certain density stays always on its isopycnal surface, hence it has to follow the up and downs of that surface. Here it comes out quite clearly that this variation in geopotential induces vertical motion w_1 (see eqn. 6.3.17) i.e. vortex tube stretching.

A good, but not concise, account.

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(4) As it was mentioned in the introduction the three dependent variables (u_0, v_0, p_0) can be expressed in terms of the pressure perturbation p_0 alone. Once p_0 is found as $p_0(x, y, z, t)$ the geostrophic equations for a stratified fluid can be ~~sole~~ used to find the quantities $(u_0, v_0, \cancel{p})$ and the hydrostatic equation is used to find ρ_0. These set of diagnostic equation is solved trivially? once the vorticity equation is solved.
Density variations eventually remove the two-dimensional character of barotropic quasi geostrophic flow inducing vertical variability. This is expressed in the geostrophic equations which can be rewritten in terms of vertical shears utilizing the hydrostatic equation. The resulting set of equations are the thermal wind or (Gill; 1982) relative geostrophic currents. They are diagnostic relationship of vertical geostrophic current shears and the horizontally varying density field:

$$\frac{\partial u_0}{\partial z} = \frac{\partial \rho_0}{\partial y} \quad , \quad \frac{\partial v_0}{\partial z} = -\frac{\partial \rho_0}{\partial x} \qquad \text{Pedlosky (6.8.7.)}$$

or dimensionally

$$f\frac{\partial u}{\partial z} = \frac{g}{\rho}\left(\frac{\partial \rho}{\partial y}\right)_p \quad , \quad f\frac{\partial v}{\partial z} = -\frac{g}{\rho}\left(\frac{\partial \rho}{\partial x}\right)_p \qquad \text{Gill (7.7.9)}$$

"...where the derivatives on the right hand side are taken on constant-pressure surface..." Gill (1982, p. 218), i.e. a particle moving on its isopycnal. All velocities are first order geostrophic velocities

(4ff)

The continuity equation to this order of approximation $O(1)$ is still degenerated as

$$\frac{1}{\rho_s}\frac{\partial}{\partial z}(w_0\rho_s) + \frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0$$

Pedlosky (6.3.7)

But the geostrophic velocities are nondivergent:

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} = 0$$

therefore

$$\frac{\partial}{\partial z} w_0 \rho_s = 0 \quad \text{also}$$

and because w_0 vanishes at the top and the bottom $w_0 = 0 = w_0(z)$

We still have a streamfct which is — in essence — the pressure perturbation p_0.

Another diagnostic equation ~~for~~ known p_0, u_0, v_0, ρ_0 can give us the $O(\varepsilon)$ vertical velocity field ~~so~~ in the absence of thermal heating:

$$-\frac{d_0\rho_0}{dt} + w_1 S = 0 \qquad (6.8.3)$$

All quantities are known except w_1.

It is noted in passing that the thermal wind equations show that vortex tube tilting might be important too. The vertical ~~velocity~~ shear of the (geostrophic) horizontal velocities induces vorticity not in the vertical but in the horizontal too. But it doesn't enter the $O(1)$ vort. dynamics.

However, the thermal wind equations state that horizontal density variations induce vertical current shear and possibly current vector rotation ~~also~~ with depth.

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(5) A particle of fluid approaching the Gulf Stream from the subtropical gyre has a certain amount of potential vorticity which it obtained from the wind stress curl. The major (on gyre scale) vorticity balance in the interior is the Sverdrup balance, i.e. wind stress curl is balanced by planetary vorticity in a ~~baro~~ tropic ocean. The length scale for such processes are the barotropic Rossby radius. L is the scale of the WSC

The dynamical lengthscale for which the dynamics represented by eqn. (3.8.9) hold is the baroclinic Rossby radius L_D which is usually much smaller than the barotropic radius (as $L_D \ll L$ in the ocean). Therefore, I think I can assume isopycnals which ~~are~~ ~~flat~~ do not vary along latitudes to enter the Gulf Stream

OK region. The lengthscale of separation of the two regimes is very different, so (clearly) variations on the large (gyre) scale are not seen

OK or negligible on the small (meso, or eddy) scale.

Let's consider for a moment the steady version of $\frac{d_0}{dt}$ which is $u_0 \frac{\partial}{\partial x} + v_0 \frac{\partial}{\partial y}$ to simplify the argument. In the high velocity regions of the Gulf Stream the relative vorticity (which was

not in our QG theory small for the particles comming from the subtropical gyre) is large when compared to the planetary vorticity. The only term which

Spin up (6.8.9) can possibly ~~support~~ balance (apart of friction) the relative vorticity then is vortex tube stretching due to changes of the upper layer depth

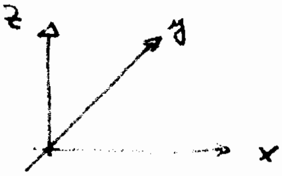
OK or strongly sloping isopycnals. The particles approaching the Gulf Stream ~~abt~~ gains anticyclonic relative vorticity ξ which is negative vorticity.

To balance this negative vorticity tendency the stretching term $-\frac{\partial}{\partial z}\rho_0$ has to provide the necessary positive vorticity tendency

that is for the moving fluid column $-\frac{\partial \rho_0}{\partial z} > 0$ & $\frac{\partial \rho_0}{\partial z} < 0$

$\frac{\partial \rho_0}{\partial z}$ has to ~~decrease~~ increase in magnitude as in a stably stratified fluid $\frac{\partial \rho}{\partial z} < 0$

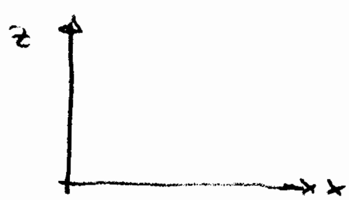
Because of $\frac{\partial v_0}{\partial z} = -\frac{\partial \rho_0}{\partial x}$



is larger than zero:

$$\frac{\partial v_0}{\partial z} > 0$$

$\frac{\partial \rho_0}{\partial x} < 0$ ✓, hence the density increases from the east toward the west ✓



✓

The relative vorticity which is small for the ✓ fluid column entering the gulf stream ~~in sum~~ increases and enters the vorticity balance. This anti-cycl. relative vorticity is balanced by vortex tube ~~stretching~~ squashing ✓ due to the rising isopycnals. Vertical velocities, therefore, are relatively large $O(\varepsilon)$. Another interpretation is that available potential energy from the subtropical gyre is released in form of kinetic energy because of the rise of the isopycnal surfaces. ✓ good!

Adding the time dependence the qualitative result of waves, instabilities and eddies due to this transfer of vorticity is anticipated. This, also, is indicated by the scale for which these dynamics hold: the baroclinic Rossby radius of deformation which corresponds closely to the eddy scale in the ocean.

In brief: In a stratified ocean which over most of its interior is in Sverdrups balance the ocean side of the Gulf Stream can be explained qualitatively by a vorticity transfer from the density field to the flow field, i.e. vortex-tube squashing due to rising isopycnals can (partly) provide ~~vorticity~~ relative vorticity present in the Gulf Stream. ✓

Very well done!

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(6) A simplified first version of

$$\frac{d_0}{dt}\left[\xi_0 + \beta y - \frac{1}{S}\frac{\partial \rho_0}{\partial z}\right] = 0$$

which is appropriate to the interior of the subtropical gyre is containing the steady part of it

$$u_0 \frac{\partial}{\partial x}\left(\xi_0 + \beta y - \frac{1}{S}\frac{\partial \rho_0}{\partial z}\right) + v_0 \frac{\partial}{\partial y}\left(\xi_0 + \beta y - \frac{1}{S}\frac{\partial \rho_0}{\partial z}\right) = 0$$

? why steady ?

or

$$u_0 \frac{\partial}{\partial x}\left(\xi_0 - \frac{1}{S}\frac{\partial \rho_0}{\partial z}\right) + v_0 \beta + v_0 \frac{\partial}{\partial y}\left(\xi_0 - \frac{1}{S}\frac{\partial \rho_0}{\partial z}\right) = 0$$

The relative vorticity is small compared to the planetary vorticity and also on the mesoscale. ~~out~~ in the interior of the gyre leaving

$$\frac{u_0}{S}\frac{\partial^2}{\partial z \partial x}\rho_0 + \frac{v_0}{S}\frac{\partial}{\partial z \partial y}\rho_0 = v_0 \beta$$

The correct statement is that the gradient of S_0 is small compared to that of the plan. vort., $\nabla S_0 << \beta$ or $\beta >> 1$.

$$u_0 \frac{\partial^2 \rho_0}{\partial z \partial x} + v_0 \frac{\partial^2 \rho_0}{\partial z \partial x} = v_0 \frac{\beta}{S}$$

$$\frac{d_0}{dt}\frac{\partial \rho_0}{\partial z} = v_0 \frac{\beta}{S} \quad \text{or} \quad \frac{d_0}{dt}\left(\beta y - \frac{1}{S}\frac{\partial \rho_0}{\partial z}\right) = 0$$

I have never seen such a balance, because the vortex tube stretching in the ocean's interior is usually provided by the Ekman pumping from an Ekman layer forced by the wind stress curl. Here the planetary vorticity is balanced by vortex tube stretching, etc, and due to sloping isopycnals. The vertical "pumping" velocity $w_1 = O(\varepsilon)$ would drive such a stratified, unforced ocean. Well, unforced is not quite the right word as   This is just the interior variation. The net effect is still governed by WSC.

or forced

input a possible forcing from the boundary now is differential heating and cooling at the surface. And indeed this is the case as the or WSC

OK tropics receive more sunlight, heat than the mid latitudes. This causes sloping North-South isopycnals. Subduction of density layers and particles or fluid columns on them would eventually be

OK described in this simplified ~~version~~ (i.e. steady) version of (6.5.8).

Note that the buoyancy pumping term scales with S^{-1} where

$$S = \left(\frac{L_D}{L}\right)^2$$

, but on the gyre scale $L_D \ll L$, hence ~~S~~

S is no longer $O(1)$. rather $S \ll 1$. Thus rather than (6.8.9) equation (6.8.14) is to be used.

no, your eq. is the one.

Another interpretation of S^{-1} is that it is the ratio between available potential energy to kinetic energy, i.e., $S = O(1)$ gives the length scale L on which ~~all~~ ~~our~~ APE associated with the sloping isopycnals is ~~transformed into~~ equally partitioned with kinetic energy.

In the subtropical gyre we found that $S = \left(\frac{L_D}{L}\right)^2 \ll 1$

~~which concludes that only a very small amount of~~

I conclude that only a very small part of the total energy is kinetic

$$S^{1/2} = \frac{L_i}{L} = \frac{\text{kinetic energy}}{\overline{APE}}$$

The APE energy is interpreted as the part of the potential energy (stored in the density, mass distribution) which can be transformed into kinetic energy supporting motion due to sloping isopycnals. As Gill (1982, p. 220) pointed out "... potential energy is hard to extract ~~from~~ in a rotating frame."

However, the ~~to~~ vorticity balance given in this section and its energy considerations (low kinetic energy content) probably resembles some features of the thermohaline circulation I speculate.

Also the wind driven part. (See Ped., Chap. 6, on large scale circ.)

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