Thermodynamics (Gill and Pedlosky)
41–43 329–331
11–13

1^{st} law $$\frac{de}{dt} + p\frac{d\rho^{-1}}{dt} = \frac{k}{\rho}\nabla^2 T + \chi + Q$$

internal energy — mechanical work done — heat diffusion (conduction) — viscous dissipation — internal heating sources

2^{nd} law $$T\,ds = de + p\,d\rho^{-1}$$
heat content

e.g. $$T\frac{ds}{dt} = \frac{de}{dt} + p\frac{d\rho^{-1}}{dt}$$

ds – differential of specific entropy
de – differential of internal energy
$d\rho^{-1}$ – differential of volume per unit mass

$s = s(T, p)$ state variables

(1) fixed volume

specific heat $$C_v \equiv T\left(\frac{\partial s}{\partial T}\right)_v = \left(\frac{\partial e}{\partial T}\right)_v + p\cancel{\left(\frac{\partial \rho^{-1}}{\partial T}\right)_v}$$

0 as ρ^{-1} is volume / unit mass

(2) fixed pressure

specific heat $$c_p \equiv T\left(\frac{\partial s}{\partial T}\right)_p = \left(\frac{\partial e}{\partial T}\right)_p + p\left(\frac{\partial \rho^{-1}}{\partial T}\right)_p$$

(3) fixed temperature

$$T\left(\frac{\partial s}{\partial p}\right)_T = \left(\frac{\partial e}{\partial p}\right)_T + p\left(\frac{\partial \rho^{-1}}{\partial p}\right)_T$$

$$\frac{\partial}{\partial p}(2): \qquad 0 = 0 + \left(\frac{\partial \rho^{-1}}{\partial T}\right)_p$$

$$\frac{\partial}{\partial T}(3): \qquad \left(\frac{\partial s}{\partial p}\right)_T = 0 + 0$$

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what type of operation is this? ~~Why not add?~~

$$-\left(\frac{\partial s}{\partial p}\right)_T = +\left(\frac{\partial \rho^{-1}}{\partial T}\right)_p$$

$$T \cdot ds \underset{s=s(p,T)}{=} T\left(\frac{\partial s}{\partial p}\right)_T dp + T\left(\frac{\partial s}{\partial T}\right)_p dT$$

$$= T(-)\left(\frac{\partial \rho^{-1}}{\partial T}\right)_p dp + c_p \quad dT$$

or $$ds = \frac{c_p}{T} dT - \left(\frac{\partial \rho^{-1}}{\partial T}\right)_p dp$$

2^{nd} law rewritten in more practical form

$$T\frac{ds}{dt} \underset{2^{nd}\ law}{=} c_p \frac{dT}{dt} - T\left(\frac{\partial \rho^{-1}}{\partial T}\right)_p \frac{dp}{dt} \underset{1^{st}\ law}{=} \frac{k}{\rho} \nabla^2 T + Q$$

<u>ideal gas</u> $\rho = \frac{p}{RT}$ $\to \rho^{-1} = \frac{RT}{p}$ $R = C_p - C_v$

$$\Rightarrow \left(\frac{\partial \rho^{-1}}{\partial T}\right)_p = \frac{R}{p} \Rightarrow T\left(\frac{\partial \rho^{-1}}{\partial T}\right)_p = \frac{RT}{p} = \frac{1}{\rho}$$

aside:

$$\Rightarrow ds = \frac{c_p}{T} dT - \frac{R}{p} dp \quad \Big| \int$$

$$s = c_p \ln T - R \ln p$$ specific entropy for ideal gas

$$T \frac{ds}{dt} \overset{2^{nd}\text{ law}}{=} c_p \frac{dT}{dt} - \frac{1}{\rho}\frac{dp}{dt} \overset{1^{st}\text{ law}}{=} \frac{k}{\rho}\nabla^2 T + Q$$

heat content — heat conduction — internal heating

$$= \frac{1}{\theta}\frac{d\theta}{dt} \cdot c_p T$$ where $\theta \equiv T\left(\frac{p_0}{p}\right)^{R/c_p}$

potential temperature

adiabatic (constant entropy) transition has $\frac{ds}{dt} = 0$ — no conduction, no internal heating

$$\Rightarrow \frac{d\theta}{dt} = 0 \quad \text{or} \quad c_p \frac{dT}{dt} = \frac{1}{\rho}\frac{dp}{dt}$$

$$\Theta = T\left(\frac{p_0}{p}\right)^{R/c_p}$$

ideal gas law $\rho = \frac{p}{RT}$

$$\frac{\rho}{p_0} = \frac{p}{p_0}\frac{1}{RT}$$

$$= \left(\frac{p_0}{p}\right)^{-1} \cdot \frac{1}{T} \cdot \frac{1}{R}$$

$$\frac{\Theta \cdot \rho}{p_0} = \cancel{T}\left(\frac{p_0}{p}\right)^{R/c_p} \Big/ \cancel{T}\left(\frac{p_0}{p}\right)^{1} \cdot \frac{1}{R}$$

$$= \frac{1}{R}\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}-1} \quad = \frac{1}{R}\left(\frac{p_0}{p}\right)^{\frac{c_p-c_v}{c_p}-1}$$ but $R = c_p - c_v$

$$= \frac{1}{R}\left(\frac{p_0}{p}\right)^{1-\frac{c_v}{c_p}-1} \quad = \frac{1}{R}\left(\frac{p_0}{p}\right)^{-c_v/c_p} \quad = \left(\frac{p}{p_0}\right)^{1/\gamma}\frac{1}{R}$$ $\gamma = \frac{c_p}{c_v}$

So $$\rho = \frac{p_0}{\Theta R}\left(\frac{p}{p_0}\right)^{1/\gamma}$$ $\rho = \rho(p, \Theta)$ state variables

$$\Delta \rho_A = \frac{p_0}{\Theta \cdot R}\frac{1}{\gamma}\left(\frac{p}{p_0}\right)^{1/\gamma - 1}\frac{dp}{p_0\, dz}\Delta z \cdot \frac{p}{p}$$ change in density of parcel raised from A to B

$$\Delta \rho_A = \frac{1}{\gamma}\underbrace{\frac{p_0}{R\Theta}\left(\frac{p}{p_0}\right)^{1/\gamma}}_{=\rho}\frac{dp}{dz}\frac{\Delta z}{p}$$

$$\Delta \rho_A = \frac{\rho}{\gamma}\frac{dp}{dz}\frac{\Delta z}{p}$$

$$\rho_A + \Delta\rho_A = \rho_A(z) + \frac{1}{\gamma}\frac{\rho}{p}\frac{\partial p}{\partial z}\Delta z$$

density of parcel initially at A raised to B

$$\rho_B = \rho_A(z) + \frac{\partial\rho}{\partial z}\Delta z$$

density of parcel at B in terms of undisturbed density parcel A had at z

$$\therefore \left(\rho_A + \Delta\rho_A\right) - \rho_B = \frac{1}{\gamma}\frac{\rho}{p}\frac{\partial p}{\partial z}\Delta z - \frac{\partial\rho}{\partial z}\Delta z$$

excess density of parcel A at its new location B

$$\frac{g}{\rho}\left[\rho_A + \Delta\rho_A - \rho_B\right] = g\underbrace{\left[\frac{1}{\gamma}\cdot\frac{1}{p}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{\partial\rho}{\partial z}\right]}_{}\Delta z$$

restoring force

$$= \frac{1}{\theta}\frac{\partial\theta}{\partial z} \quad !$$

2 pages of algebra (attached) 6+7

$$= \frac{g}{\theta}\frac{\partial\theta}{\partial z}\Delta z$$

restoring force

$$= N^2 \Delta z$$

$$N = \sqrt{\frac{g}{\theta}\frac{\partial\theta}{\partial z}}$$

also $$\frac{1}{\theta}\frac{\partial\theta}{\partial z} = \frac{1}{T}\frac{\partial T}{\partial z} - \frac{R}{c_p}\frac{1}{p}\underbrace{\frac{\partial p}{\partial z}}_{\text{hydrostatic}}$$

same 2 pages of algebra

$$= \frac{1}{T}\frac{\partial T}{\partial z} - \frac{R}{c_p}\frac{1}{p}\overbrace{(-)\,\underbrace{\rho}_{\text{ideal gas}}\cdot g}^{}\,\frac{T}{T}$$

$$= \frac{1}{T}\frac{\partial T}{\partial z} + \frac{R\cancel{T}}{\cancel{p}}\cdot\frac{g}{c_p}\cdot\frac{\cancel{p}}{\cancel{RT}}\cdot\frac{1}{T} = \frac{1}{T}\left(\frac{\partial T}{\partial z} + \frac{g}{c_p}\right)$$

$$\theta = T\left(\frac{p_0}{p}\right)^{R/c_p}$$

$$\rho = \frac{p_0}{R\theta}\left(\frac{p}{p_0}\right)^{1/\gamma} \qquad\qquad \rho = \frac{p}{RT}$$

$$\frac{\partial\theta}{\partial z} = \frac{\partial T}{\partial z}\left(\frac{p_0}{p}\right)^{R/c_p} + T\,\frac{R}{c_p}\left(\frac{p_0}{p}\right)^{R/c_p - 1}\cdot(-)\,p_0\,\frac{\partial p}{\partial z}\cdot\frac{1}{p^2}$$

$$= \frac{\partial T}{\partial z}\left(\frac{p_0}{p}\right)^{R/c_p} + \frac{RT}{c_p}\left(\frac{p_0}{p}\right)^{R/c_p}(-)\,\frac{1}{p}\,\frac{\partial p}{\partial z}$$

$$= \frac{\partial T}{\partial z}\left(\frac{p_0}{p}\right)^{R/c_p} - \frac{1}{\rho c_p}\left(\frac{p_0}{p}\right)^{R/c_p}\frac{\partial p}{\partial z}$$

$$\frac{1}{\theta}\frac{\partial\theta}{\partial z} = \frac{\frac{\partial T}{\partial z}\cancel{\left(\frac{p_0}{p}\right)^{R/c_p}}}{T\cancel{\left(\frac{p_0}{p}\right)^{R/c_p}}} - \frac{\frac{1}{\rho c_p}\cancel{\left(\frac{p_0}{p}\right)^{R/c_p}}\frac{\partial p}{\partial z}}{T\cancel{\left(\frac{p_0}{p}\right)^{R/c_p}}}$$

$$= \frac{1}{T}\frac{\partial T}{\partial z} - \frac{1}{\rho c_p T}\frac{\partial p}{\partial z}$$

$$= \frac{1}{T}\frac{\partial T}{\partial z} - \frac{R\theta}{p_0 (p/p_0)^{1/\gamma}}\cdot\frac{1}{c_p T}\frac{\partial p}{\partial z}$$

$$\frac{R}{c_p} + \frac{1}{\gamma} = \frac{c_p - c_v}{c_p} + \frac{c_v}{c_p} = 1 - \frac{c_v}{c_p} + \frac{c_v}{c_p} = 1$$

$$\left(\frac{p_0}{p}\right)^{R/c_p + \frac{1}{\gamma}}$$

$$= \frac{1}{T}\frac{\partial T}{\partial z} - \frac{R\cancel{T}\left(\frac{p_0}{p}\right)^{R/c_p}}{p_0\,(p_0/p)^{-1/\gamma}}\cdot\frac{1}{c_p\cancel{T}}\frac{\partial p}{\partial z}$$

$$\boxed{\frac{1}{\theta}\frac{\partial \theta}{\partial z} = \frac{1}{T}\frac{\partial T}{\partial z} - R \cdot \frac{\cancel{p_0}}{\cancel{p_0}} \cdot \frac{1}{p} \frac{1}{c_p}\frac{\partial p}{\partial z}}$$

$$\frac{R}{c_p} = \frac{c_p - c_v}{c_p} = \frac{1 - c_v}{c_p} = 1 - \frac{1}{\gamma}$$

$$= \frac{1}{T}\frac{\partial T}{\partial z} - \frac{1}{p}\frac{\partial p}{\partial z} + \frac{1}{\gamma}\frac{1}{p}\frac{\partial p}{\partial z}$$

$$= \frac{1}{T}\frac{\partial T}{\partial z} - \left(\frac{1}{p}\frac{\partial p}{\partial z} + \frac{1}{\gamma}\frac{1}{p}\frac{\partial p}{\partial z}\right)$$

$$= \frac{1}{T}\frac{\partial T}{\partial z} - \frac{1}{p}\frac{\partial p}{\partial z}\left(1 - \frac{1}{\gamma}\right)$$

$$= \frac{1}{T}\frac{\partial T}{\partial z} - \frac{1}{p}\frac{\partial p}{\partial z} + \frac{1}{\gamma}\frac{1}{p}\frac{\partial p}{\partial z}$$

$$= \cancel{\frac{1}{p}\frac{\partial p}{\partial z}} - \frac{1}{\rho}\frac{\partial \rho}{\partial z} - \cancel{\frac{1}{p}\frac{\partial p}{\partial z}} + \frac{1}{\gamma}\frac{1}{p}\frac{\partial p}{\partial z} = -\frac{1}{\rho}\frac{\partial \rho}{\partial z} + \frac{1}{\gamma}\frac{1}{p}\frac{\partial p}{\partial z}$$

$$\rho = \frac{p}{R \cdot T} \qquad R = c_p - c_v \qquad p = \rho \cdot R \cdot T$$

$$\frac{1}{R}\frac{\partial p}{\partial z} = \rho\frac{\partial T}{\partial z} + T\frac{\partial \rho}{\partial z}$$

$$T = \frac{p}{R \cdot \rho}$$

$$\frac{1}{R}\frac{\partial p}{\partial z} = \rho\frac{\partial T}{\partial z} + T\frac{\partial \rho}{\partial z}$$

$$R\frac{\partial T}{\partial z} = \left(\frac{\partial p}{\partial z} \cdot \rho - p\frac{\partial \rho}{\partial z}\right) \cdot \frac{1}{\rho^2} = \frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{p}{\rho^2}\frac{\partial \rho}{\partial z}$$

$$\cancel{R}\,\frac{1}{T}\frac{\partial T}{\partial z} = \cancel{p}\frac{\cancel{R\rho}}{p\,\cancel{\rho}}\frac{1}{}\frac{\partial p}{\partial z} - \frac{\cancel{p}}{\rho^2}\frac{\partial \rho}{\partial z} \cdot \frac{\cancel{R\rho}}{\cancel{p}}$$

$$\frac{1}{T}\frac{\partial T}{\partial z} = \frac{1}{p}\frac{\partial p}{\partial z} - \frac{1}{\rho}\frac{\partial \rho}{\partial z}$$