1. The linearized shallow water equations on an f-plane of constant depth H

\[
\begin{align*}
    u_t - fv + g \eta_x &= 0 \\
    v_t + fu + g \eta_y &= 0 \\
    \eta_t + Hu_x + Hv_y &= 0
\end{align*}
\]

allow waves of the form

\[
(u, v, \eta) = (u_0, v_0, \eta_0) \exp[i(kx + ly - \omega t)]
\]

where \((k, l)\) are wave numbers in \((x, y)\) directions, \(\omega\) is a wave frequency, \(i = \sqrt{-1}\), and subscripts denote differentiations. This linear problem can also be written in matrix form as \(A \cdot (u, v, \eta) = 0\) where \(A\) is a 3x3 matrix of constant coefficients which for nontrivial solutions has a determinant \(\det(A) = 0\).

(a) Find the dispersion relation for this class of waves by exploiting \(\det(A) = 0\). \([5\text{pts}]\)

(b) Find the solutions for velocity \((u, v)\) and show that the velocity vectors describe ellipses with a ratio of minor to major axes is \(f/\omega\). [Hint: The problem is greatly simplified by choosing a co-ordinate system oriented in the direction of wave propagation, e.g., assume \(\eta = \eta_0 \cos(kx - \omega t)\).] \([10\text{pt}]\)

(c) Discuss the differences of horizontal current ellipses in the long- and short wave limits, e.g., for \(\kappa a \ll 1\) and \(\kappa a \gg 1\) where \(a\) is Rossby radius of deformation \(a = (\sqrt{gH})/f\) and \(\kappa = \sqrt{(k^2 + l^2)}\). Specifically, comment on the sense of current rotation and ellipticities. \([5\text{pts}]\)

2. A wave has the dispersion relation

\[
\omega = -\beta_0 a^2 k / [1 + a^2 (k^2 + l^2)]
\]

where \(a = \text{const.}\) is the Rossby radius of deformation, \(\beta_0\) is the spatial gradient of the Coriolis parameter \(f\), and \((k, l)\) are wavenumbers in the east-west, north-south direction, and \(\omega\) is the wave frequency:

(a) Find the phase velocities of these waves. \([2\text{pts}]\)

(b) Find the group velocities for these waves. \([3\text{pts}]\)

(c) Compare and contrast the phase and group velocities in the short and long wave limits. \([5\text{pts}]\)

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